Pricing Algorithms for financial derivatives on baskets modeled by Lévy copulas

Christoph Winter, ETH Zurich, Seminar for Applied Mathematics

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Introduction

Option pricing
  Partial integrodifferential equation
  Variational formulation

Discretization
  Sparse tensor product finite element space
  Galerkin discretization
  Numerical quadrature of the Lévy copula kernel

Numerical results

Conclusion
Literature


Lévy copula and tail integral

A function $F : \overline{\mathbb{R}}^d \to \overline{\mathbb{R}}$ is called Lévy copula if

- $F(u_1, \ldots, u_d) \neq \infty$ for $(u_1, \ldots, u_d) \neq (\infty, \ldots, \infty)$,
- $F(u_1, \ldots, u_d) = 0$ if $u_i = 0$ for at least one $i \in \{1, \ldots, d\}$,
- $F$ is $d$-increasing,
- $F^{\{i\}}(u) = u$ for any $i \in \{1, \ldots, d\}$, $u \in \mathbb{R}$.

The tail integral $U : \mathbb{R}^d \setminus \{0\} \to \mathbb{R}$

$$U(x_1, \ldots, x_2) = \prod_{i=1}^d \text{sgn}(x_j)^\nu \left( \prod_{j=1}^d I(x_j) \right).$$
Theorem (Sklar’s theorem for Lévy copulas)

For any Lévy process $X \in \mathbb{R}^d$ exists a Lévy copula $F$ such that the tail integrals of $X$ satisfy

$$U^I ((x_i)_{i \in I}) = F^I ((U_i(x_i))_{i \in I}),$$

for any nonempty $I \subset \{1, \ldots, d\}$ and any $(x_i)_{i \in I} \in \mathbb{R}^{|I|} \setminus \{0\}$. The Lévy copula $F$ is unique on $\prod_{i=1}^d \text{Ran} U_i$.

Lévy density $k$ with marginal Lévy densities $k_1, \ldots, k_d$,

$$k(x_1, \ldots, x_d) = \partial_1 \cdots \partial_d F|_{\xi_1 = u_1(x_1), \ldots, \xi_d = u_d(x_d)} k_1(x) \cdots k_d(x).$$
Clayton Lévy copula

In two dimensions (d=2)

\[
F(u, v) = \left( |u|^{-\theta} + |v|^{-\theta} \right)^{-\frac{1}{\theta}} \left( \eta 1_{\{uv \geq 0\}} - (1 - \eta) 1_{\{uv \leq 0\}} \right),
\]

\((\alpha_1, \alpha_2)\)-stable marginal densities

\[
k(x_1, x_2) = (1 + \theta) \alpha_1^{\theta+1} \alpha_2^{\theta+1} |x_1|^{\alpha_1 \theta - 1} |x_2|^{\alpha_2 \theta - 1}
\]

\[
\cdot \left( \alpha_1^{\theta} |x_1|^{\alpha_1 \theta} + \alpha_2^{\theta} |x_2|^{\alpha_2 \theta} \right)^{-\frac{1}{\theta} - 2} \left( \eta 1_{\{x_1 x_2 \geq 0\}} + (1 - \eta) 1_{\{x_1 x_2 < 0\}} \right).
\]
Clayton Lévy copula with marginal of CGMY type

\[ C_i = 1, \ G_i = M_i = 4, \ Y_i = 1 \text{ for } i = 1, 2 \text{ and } \eta = \frac{1}{2} \]

Independent \( \theta = 0.5 \) (left) and dependent \( \theta = 10 \) (right) tails.
Tempered stable Lévy copula processes

Let densities $k_1, \ldots, k_d$ be tempered stable.

With Sklar’s theorem for Lévy copulas there exist a Lévy process $X_t \in \mathbb{R}^d$ with marginal densities $k_1, \ldots, k_d$.

Log prices are solution of the generalized BS equation

$$\frac{\partial u}{\partial t} + \mathcal{A}u = 0, \quad u|_{t=T} = g,$$

where $\mathcal{A}$ is the infinitesimal generator of the process $X_t$ with domain $\mathcal{D}(\mathcal{A})$. 
Partial integrodifferential equation

Assume \( S_t^i = S_0^i e^{rt + X_t^i} \), 1 \( \leq i \leq d \). The price

\[
V(t, S) = \mathbb{E} \left( e^{-r(T-t)} g(S_T) | S_t = S \right),
\]

is the solution of

\[
\frac{\partial V}{\partial t}(t, S) + \frac{1}{2} \sum_{i,j=1}^{d} S_i S_j A_{ij} \frac{\partial^2 V}{\partial S_i \partial S_j} + r \sum_{i=1}^{d} S_i \frac{\partial V}{\partial S_i}(t, S) - rV(t, S)
\]

\[
+ \int_{\mathbb{R}^d} \left( V(t, Se^Z) - V(t, S) - \sum_{i=1}^{d} S_i (e^{Z_i} - 1) \frac{\partial V}{\partial S_i}(t, S) \right) \nu(dZ) = 0.
\]

Terminal condition \( V(T, S) = g(S) \).
Transformation to log price

Let \( x_i = \log S_i, \tau = T - t \).

\[
\frac{\partial u}{\partial \tau} + \mathcal{A}_{BS}[u] + \mathcal{A}_{J}[u] = 0 ,
\]

with

\[
\mathcal{A}_{BS}[\varphi] = -\frac{1}{2} \sum_{i,j=1}^{d} A_{ij} \frac{\partial^2 \varphi}{\partial x_i \partial x_j} + \sum_{i=1}^{d} \left( \frac{1}{2} A_{ii} - r \right) \frac{\partial \varphi}{\partial x_i} + r \varphi ,
\]

\[
\mathcal{A}_{J}[\varphi] = - \int_{\mathbb{R}^d} \left( \varphi(x + z) - \varphi(x) - \sum_{i=1}^{d} (e^{z_i} - 1) \frac{\partial \varphi}{\partial x_i}(x) \right) \nu(dz) .
\]

Initial condition

\[
u(0, x) := u_0 = g(e^{x_1}, \ldots, e^{x_d}) .
\]
Variational formulation

Basket option $g(e^{x_1}, e^{x_2}, \ldots, e^{x_d}) = \left(1 - \sum_{i=1}^{d} e^{x_i}\right)^+$.  

Weighted Sobolev space

$$H^1_\eta(\mathbb{R}^d) := \left\{ \varphi \in L^1_{\text{loc}}(\mathbb{R}^d) \mid e^{\eta(x)} \varphi, e^{\eta(x)} \frac{\partial \varphi}{\partial x_i} \in L^2(\mathbb{R}^d), i = 1, \ldots, d \right\},$$

Payoff $g \in H^1_{-\eta}(\mathbb{R}^d)$ where

$$\eta(x) = \sum_{i=1}^{d} \left( \mu_i^+ 1_{\{x_i>0\}} + \mu_i^- 1_{\{x_i<0\}} \right) |x_i|,$$

with $\mu_i^+ > 1$, $\mu_i^- > 0$.  

Bilinear forms

We associate with $\mathcal{A}_{BS}$ the bilinear form

$$a^n_{BS}(u, v) = \int_{\mathbb{R}^d} \mathcal{A}_{BS}[u](x)v(x)e^{2\eta(x)} \, dx,$$

and with $\mathcal{A}_J$

$$a^n_J(u, v) = \int_{\mathbb{R}^d} \mathcal{A}_J[u](x)v(x)e^{2\eta(x)} \, dx,$$

and set

$$a^n(u, v) = a^n_{BS}(u, v) + a^n_J(u, v).$$
Continuity and Gårding inequality

Assume $A > 0$ and $\eta \in L^1_{\text{loc}}(\mathbb{R}^d)$ satisfies

(i) $\frac{\partial \eta}{\partial x_i} \in L^\infty(\mathbb{R}^d)$, $1 \leq i \leq d$,

(ii) $\eta(x + \theta z) - \eta(x) \leq \eta(z)$, $\forall x, z \in \mathbb{R}^d$, $\forall \theta \in [0, 1]$,

(iii) $\int_{\mathbb{R}^d} e^{\eta(z)} |z| 1_{\{|z| > 1\}} \nu(dz) < \infty$.

Then, there exist constants $C_1, C_2, C_3 > 0$ such that

$|a^{-\eta}(u, v)| \leq C_1 \|u\|_{H^{-\eta}(\mathbb{R}^d)} \|v\|_{H^{-\eta}(\mathbb{R}^d)}$,

$|a^{-\eta}(u, u)| \geq C_2 \|u\|^2_{H^{-\eta}(\mathbb{R}^d)} - C_3 \|u\|^2_{L^2_{-\eta}(\mathbb{R}^d)}$. 
Sparse tensor product finite element space

d=1: Wavelet basis on \([-R, R]\)

\[ V^L = \text{span} \left\{ \psi_j^\ell \mid 0 \leq \ell \leq L, \ 1 \leq j \leq M^\ell \right\} = \mathcal{W}^0 \oplus \cdots \oplus \mathcal{W}^\ell, \]

with increment spaces

\[ \mathcal{W}^0 := \mathcal{V}^0, \quad \mathcal{W}^\ell := \text{span} \left\{ \psi_j^\ell : 1 \leq j \leq M^\ell \right\}. \]

In \([-R, R]^d\)

\[ V^L := V^L \otimes \cdots \otimes V^L = \bigoplus_{0 \leq \ell_i \leq L} \mathcal{W}^{\ell_1} \otimes \cdots \otimes \mathcal{W}^{\ell_d}. \]

Sparse tensor product space

\[ \widehat{V}^L := \bigoplus_{0 \leq \ell_1 + \cdots + \ell_d \leq L} \mathcal{W}^{\ell_1} \otimes \cdots \otimes \mathcal{W}^{\ell_d}. \]
Sparse tensor product space \((d = 2)\)

Difference between \(V^L\) and \(\hat{V}^L\) for level \(L = 3\).
Galerkin discretization

Ansatz

\[ u^L(\tau, x) = \sum_{\ell,j} u^L_j(\tau) \psi^L_j(x). \]

Linear system

\[ M^d U'(\tau) + A^d U(\tau) = 0, \quad \forall \tau \in (0, T), \]
\[ U(0) = U_0. \]

Backward Euler time stepping

\[ \left( M^d + \Delta t A^d \right) U(\tau_m) = M^d U(\tau_{m-1}), \quad m = 1, \ldots, M, \]
\[ U(0) = U_0. \]
Discretized operator \((d = 2)\)

We need to compute

\[
da_{BS}(\psi_j^\ell, \psi_{j'}^\ell') = \sum_{i,k=1}^{d} \frac{1}{2} A_{ik} \int_{\Omega_R} \frac{\partial \psi_j^\ell}{\partial x_i} \frac{\partial \psi_{j'}^\ell'}{\partial x_k} \, dx + \sum_{i=1}^{d} \frac{1}{2} A_{ii} \int_{\Omega_R} \frac{\partial \psi_j^\ell}{\partial x_i} \psi_{j'}^\ell' \, dx.
\]

In matrix form

\[
A_{BS}^2 := \frac{1}{2} A_{11} S \hat{\otimes} M + \frac{1}{2} A_{22} M \hat{\otimes} S + A_{12} C \hat{\otimes} (-C) \\
+ \frac{1}{2} A_{11} C \hat{\otimes} M + \frac{1}{2} A_{22} M \hat{\otimes} C.
\]
Discretized operator \((d = 2)\)

For the jump part

\[
\mathbf{a}_j(\psi_j^l, \psi_j^{l'}) = - \int_{\mathbb{R}^d} \int_{\Omega_R} \left( \psi_j^l(x + z) \psi_j^{l'}(x) - \psi_j^l(x) \psi_j^{l'}(x) \right.
\]

\[
- \sum_{i=1}^{d} \left( e^{z_i} - 1 \right) \frac{\partial \psi_j^l}{\partial x_i} \psi_j^{l'} \, dx \bigg) \nu(dz) .
\]

In matrix form

\[
\mathbf{A}_j^2 := - \int_{\mathbb{R}^d} \left( \mathbf{M}^{0,z_1} \otimes \mathbf{M}^{0,z_2} - \mathbf{M} \otimes \mathbf{M} 
\]

\[
- (e^{z_1} - 1) \mathbf{C} \otimes \mathbf{M} - (e^{z_2} - 1) \mathbf{M} \otimes \mathbf{C} \bigg) \nu(dz) .
\]
Numerical quadrature of the Lévy Copula kernel

Quadrature points for $N = 6$ and $\theta = 0.5$. Computation of

$$\int_{\Omega_R} |z|^2 k(z) \, dz$$

with $C_i = 1$, $G_i = M_i = 8$, $Y_i = 1$ for $i = 1, 2$ and $\eta = 1$. 
Operator Matrix (in wavelet basis)

For \( d = 2, \ L = 5 \) and \( R = 5 \)
\( C = (1, 1), \ Y = (1, 1), \ G = (8, 8), \ M = (8, 8) \) and \( \eta = \frac{1}{2}, \ \theta = 10 \).
Matrix on full grid (left) and on sparse grid (right).
Comparison stable vs tempered stable processes

For level $L = 6$.

Matrix for stable processes (left) and tempered stable processes (right).
Multi-asset options

Let $T = 0.5$ and $r = 0$.

Maximum put options (left) $g = (1 - \max(S_1, S_2, \ldots, S_d))^+$

Basket options (right) $g = \left(1 - \sum_{i=1}^d S_i\right)^+$
Influence of the dependence structure

Difference between strong and weak dependence for maximum put option (left) and basket option (right)
Conclusion

- Efficient quadrature rule
- Sparse grid technique
- Influence of exponential tails on matrix
- Influence of dependence structure on option prices