Deterministic implied volatility models

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Abstract
In this paper, we characterize two deterministic implied volatility models, defined by assuming that either the per-delta or the per-strike implied volatility surface has a deterministic evolution. Practitioners have recently proposed these two models to describe two regimes of implied volatility (see Derman (1999 Risk 4 55–9)). In an arbitrage-free sticky-delta model, we show that the underlying asset price is the exponential of a process with independent increments under the unique risk neutral measure and that any square-integrable claim can be replicated up to a vanishing risk by trading portfolios of vanilla options. This latter result is similar in nature to the quasi-completeness result obtained by Bjork et al (1997 Finance Stochastics 1 141–74) for interest rate models driven by Levy processes. Finally, we show that the only arbitrage-free sticky-strike model is the standard Black–Scholes model.

1. Introduction

In classic extensions of the Black–Scholes (1973) model that accounts for the smile effect, the underlying asset price \( S \) is driven by one or two Brownian motions. These models are referred as local volatility models as they differ from the Black–Scholes model by simply allowing the local volatility \( \sigma_t \) of the underlying asset price to be stochastic:

\[
dS_t/S_t = \mu_t \, dt + \sigma_t \, dW_t.
\]

Two types of local volatility models have been proposed: the so-called deterministic and stochastic local volatility models. In the deterministic local volatility models, the local volatility satisfies \( \sigma_t = \sigma(t,S_t) \) as in Dupire (1994) and Derman and Kani (1994). Deterministic local volatility models are complete and complex options are replicated using the underlying asset price to be stochastic:

\[
dS_t/S_t = \mu_t \, dt + \sigma_t \, dW_t.
\]

The processes \( B \) and \( W \) are Brownian motions with correlation \( \rho \) and are defined under the risk-neutral measure. This risk-neutral measure is unique and the stochastic local volatility models are complete if call options are traded instruments.

The parameters \( \alpha_t, \kappa, \xi \) and \( \rho \) are implied by calibration to the initial smile surface. When inferred from historical data, these parameters are typically much lower. The reason for this can be understood if we observe that in stochastic models, \( \text{d} \ln S_t \) has a normal conditional distribution with variance \( \sigma_t^2 \, dt \) as in the Black–Scholes model. Consequently, the calibration of a stochastic local volatility model to a short-dated smile curve, which typically has large variations near the at-the-money strike, results in unrealistically large correlation \( \rho \) and volatility \( \xi \). To compensate for these large parameters, the calibration to a long-dated smile curve, which is typically flatter, implies a large mean-reversion rate \( \kappa \). Deterministic local volatility models have similar problems despite accurate calibrations. The local volatility function inferred from the initial smile surface, has typically ‘unrealistically’ large slope and convexity for small maturity while it is almost flat for large maturity.
It is important to understand that a complex option will be hedged efficiently using a smile model only if the model implies a dynamic of the smile that is sufficiently ‘realistic’ and ‘stationary’ for the model to not require frequent re-calibration. Local volatility models are, in this sense, unsatisfactory. The dynamic of the implied volatility surface is entirely specified once the model has been calibrated with the initial smile surface, leaving thus no control on this dynamic.

To illustrate further this point, consider the foreign-exchange market, where implied volatilities are quoted per delta. The underlying $S$ is typically traded more frequently than vanilla options and thus, the implied volatilities do not change as frequently as $S$ does. In a deterministic local volatility model such as the Dupire model, the implied volatilities are however functions of $S$. Hence, over a short time interval during which implied volatilities do not change, the model will still need re-calibration every time $S$ changes! These frequent re-calibrations cause the hedging strategies implied by the model to be ‘inefficient’ as reported by Dumas et al (1998). In a stochastic local volatility model, implied volatilities depend on $t$, the local volatility $\sigma$, and on the percentage-in-the-money $S_t/K$. This last dependence implies a greater stability of the corresponding hedging strategies. However, to be consistent with market smiles, such a model over-estimates, as previously explained, the volatility of volatility and the correlation of volatility with $S$ and, as a consequence, the cost of hedging is not accurately accounted for.

The need for smile models that are calibrated to an initial smile surface and that imply a ‘realistic’ evolution of implied volatility explains the recent interest in the so-called implied volatility models. These models are defined by direct assumptions on the stochastic evolution of the smile surface from an initial surface, as in Schonbucher (1999) and in Cont and da Fonseca (2001). It is important to stress, however, that these models need severe restrictions to be arbitrage-free.

In this paper, we characterize two recently proposed deterministic implied volatility models, defined by assuming that either the per-delta or the per-strike implied volatility surface has a deterministic evolution. In arbitrage-free sticky-delta models, we show that the underlying asset price is the exponential of a process with independent increments under the risk neutral measure and that any square-integrable claim can be replicated up to a vanishing risk by trading portfolios of vanilla options. Finally, we show that the only arbitrage-free sticky-strike model is the Black–Scholes model.

2. The option market and the implied volatility models

We consider a continuous trading economy on a finite horizon $[0, T]$ with traded assets at time $t$, a financial asset $S$, the money market account $B$, the call and the put options on $S$ with strike $K > 0$ and maturity in $(t, t + \Delta)$. We also assume that static portfolios consisting of a continuum of traded vanilla options are traded instruments as in Breedan and Litzenberger (1978) and Carr and Madan (1998).

We assume that only short-dated vanilla options are traded assets because in typical option markets, only short-dated options have liquid enough prices to be regarded as traded instruments. In typical foreign exchange markets for example, only the vanilla options with maturity less than a couple of years have liquid prices.

We denote by $S_t > 0$, $C_t(x, K) > 0$ and $P_t(x, K) > 0$ the price at time $t$ of one unit of $S$, of one call and of one put option on $S$ with maturity $t + x$ in $(t, t + \Delta)$ and with strike $K > 0$.

We assume no transaction costs. We assume deterministic interest rates and we denote by $P_{t,s}$ the price at time $t$ of a discount bond with maturity $t + x$ and by $B_t \equiv 1/P_0$ the money–market account. We denote by $F_{t,x}$ the forward price at time $t$ of receiving one unit of $S$ at time $t + x \geq t$.

We assume that there exists a real bounded function $\mu(t)$ such that

$$F_{t,x}/S_t = \exp \left( \int_{t}^{t+x} \mu(s) \, ds \right) = m_{t,x}. $$

We assume that there exists a stochastic basis $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t : t \in [0, T]\})$ with a right-continuous, complete, increasing filtration with respect to which $S_t$ and $1/S_t$ are cadlag quasi-left continuous square-integrable processes and

$$ \text{sup}\{E[S_t^2 + S_{t-}^2] : 0 \leq t \leq T\} < \infty, $$

$$ C_t(x, K) = P_{t,x} E_t[(S_{t+x} - K)^+], $$

$$ P_t(x, K) = P_{t,x} E_t[(K - S_{t+x})^+]. $$

We assume that the map $(x, K) \mapsto C_t(x, K)$ from $(0, x_m) \times [0, +\infty)$ into $(0, +\infty)$ is of class $C^1$ (resp. $C^2$) in the first (resp. second) variable for all $t \in (0, T)$. The market filtration $\{\mathcal{F}_t : t \in [0, T]\}$ is the filtration generated by all primary traded asset prices.

**Definition 2.1.** A family $\{C_t(x, K), P_t(x, K), S_t : (x, K) \in (0, \infty) \times (0, +\infty)\}$ satisfying the above assumptions is called an implied volatility model.

We use the terminology of implied volatility models instead of option price models because implied volatilities rather than option prices are the financial observable (see section 1). It is clear, however, that a process $S_t$ and a two-parameter family of implied volatility processes define an implied volatility model providing that this specification is arbitrage-free. Our definition of implied volatility models is based on option prices to ensure our implied volatility models are arbitrage-free.

We shall need the following technical restriction to ensure that $S$ has bounded local characteristics. This restriction is fairly mild and met by most local volatility models.

**Definition 2.2.** An implied volatility model is regular if

(i) $\lim_{t \to 0} E[f(S_{t+})|S_t = S] = f(S), \, (f \in C_0)$,

(ii) $\sup_t \, \delta E[(S_{t+}/F_{t,x})^2|S_t] \text{ is locally bounded},$

(iii) $\sup_t E[\sup_{\Delta} |\delta E[(S_{t+}/F_{t,x})^{2\Delta}]|] < \infty$.

We write $C_0$ for the space of continuous functions vanishing at zero and infinity. We refer to Derman and Kani (1998), Schonbucher (1999) and Cont and da Fonseca (2001) for examples of stochastic implied volatility models. We now restrict our attention to two deterministic implied volatility models.
3. The sticky-delta and the sticky-strike implied volatility models

The Black–Scholes function is denoted \( C^{BS}(V, F, K) \) with
\[
C^{BS}(V, F, K) = N(d) - N(d - \sqrt{V}),
\]
where
\[
d = \ln(F/K)/\sqrt{V} + \frac{1}{2} \sqrt{V}.
\]

We denote
\[
\Delta^{BS}(V, F, K) = \partial_{F} C^{BS}(V, F, K),
\]
\[
\Gamma^{BS}(V, F, K) = \partial_{F}^{2} C^{BS}(V, F, K).
\]
We recall that
\[
\partial_{F} C^{BS}(V, F, K) = \frac{1}{2} F^{2} \Gamma^{BS}(V, F, K).
\]

**Definition 3.1.** Let \((x, I) \in (0, x_{m}) \times (0, +\infty)\). The per-delta implied volatility denoted \( \sigma_{1}(x, I) \) where \( I \) stands for the forward moneyness ratio, is the unique positive solution of the equation
\[
P_{tx} C^{BS}(x\sigma_{1}(x, I)^{2}, F_{tx}, F_{tx}) = C_{1}(x, F_{tx} I).
\]

The per-strike implied volatility denoted \( \Sigma_{1}(x, K) \) is the unique positive solution of the equation
\[
P_{tx} C^{BS}(x\Sigma_{1}(x, K)^{2}, F_{tx}, K) = C_{1}(x, K).
\]

Since the function \( C^{BS}(V, F, K) \) is strictly increasing with respect to \( V \), we conclude that
\[
\Sigma_{1}(x, K) = \sigma_{1}(x, K / F_{tx}).
\]

At a given time \( t \), the volatility functions \((x, I) \mapsto \sigma_{1}(x, I) \) and \((x, K) \mapsto \Sigma_{1}(x, K)\) are of class \( C^{1} \) in the first variable and \( C^{2} \) in the second variable in \((0, x_{m}) \times (0, +\infty)\). For \((x, I) \in (0, x_{m}) \times (0, +\infty)\), the implied volatility processes \( \sigma_{1}(x, I) \) and \( \Sigma_{1}(x, K) \) are caglad quasi-left continuous processes adapted to the market filtration.

Following Derman (1999) and Reiner (1999), we propose the following definition of sticky-delta and sticky-strike implied volatility models.

**Definition 3.2.** An implied volatility model is sticky-delta if the per-delta volatility process \( \sigma_{1}(x, I) \) is deterministic on \([0, T]\) for all \((x, I) \in (0, x_{m}) \times (0, +\infty)\).

An implied volatility model is sticky-strike if the per-strike volatility process \( \Sigma_{1}(x, K) \) is deterministic on \([0, T]\) for all \((x, K) \in (0, x_{m}) \times (0, +\infty)\).

We observe that the Black–Scholes model is an implied volatility model defined by \( \sigma_{1}(x, I) = \Sigma_{1}(x, K) = \sigma_{1}(x, I) \).

4. Characterization of the sticky-delta implied volatility models

For a sticky-delta implied volatility model, we define the following deterministic function:
\[
c(t, x, I) = C^{BS}(x\sigma_{1}(x, I)^{2}, 1, I). \tag{4.1}
\]

Since \( y^{2} < 2(e^{y} + e^{-y}) \) for all \( y \) and \( S_{t} / S_{i} \) are square-integrable, \( \ln S_{t} \) is square-integrable. Finally, we have the following result.

**Theorem 4.1.** In a sticky-delta implied volatility model defined on \([0, T]\), the market filtration \( \{ \mathcal{F}_{t} : 0 \leq t \leq T \} \) coincides with the filtration generated by \( S \), all risk-neutral probability measures are equal to the probability measure \( P \) on the \( \sigma \)-algebra \( \mathcal{F}_{T} \) and the stochastic process \( \ln S_{t} \) has independent increments under \( P \).

**Proof.** We note that \( P \) is a risk-neutral measure. Consider a forward-start call option with fixing date \( t \) and maturity \( t + x \) with \( x < x_{m} \). This option pays at maturity the quantity
\[
(S_{t+x} - k F_{tx})^{+}/F_{tx}.
\]

At time \( t \), this forward-start option is a regular call option on \( S \) with strike \( k F_{tx} \), time-to-maturity \( x \) and notional \( 1 / F_{tx} \). At time \( t \), the cost of replicating this option is \( P_{tx} c(t, x, k) \). This value is deterministic. Hence at time \( s \leq t \), the forward value of this option is \( c(t, x, k) \) and thus independent of \( s \). Finally, we have proved that for any risk-neutral measure \( Q \) and any \( s \leq t \), we have
\[
E_{Q}[(S_{t+x} - k F_{tx})^{+}/F_{tx}] = c(t, x, k). \tag{4.2}
\]

Next, we note that
\[
0 \leq \frac{1}{F_{tx}} \partial_{k}(S_{t+x} - k F_{tx})^{+} \leq 1. \tag{4.3}
\]

We can thus permute differentiation with respect to \( k \) and \( Q \)-expectation in (4.2):
\[
Pr^{Q}[S_{t+x} < k F_{tx} | \mathcal{F}_{s}] = 1 + \partial_{k} c(t, x, k). \tag{4.4}
\]

The cumulative distribution of \( \ln S_{t+x} = \ln S_{t} \) conditional on \( \mathcal{F}_{s} \) is thus independent of \( s \leq t \), \( S_{t} \) and \( Q \).

Let \( u_{1} = \ln S_{t} - \ln S_{s} \) and \( u_{2} = \ln S_{s} - \ln S_{u} \) with \( s < r < t < u < T \), \( 0 < r - u < x_{m} \) and \( u - t < x_{m} \). Equation (4.4) implies that for any risk-neutral measure \( Q \), we have
\[
E_{Q}[e^{iu_{1} + iu_{2}}] = E_{Q}[e^{iu_{1}}] E_{Q}[e^{iu_{2}}]. \tag{4.5}
\]

The variables \( u_{1} \) and \( u_{2} \) are thus independent and have the same joint distribution under any risk-neutral measure \( Q \). This result is extended by convolution to \( n \) arbitrary non-overlapping increments of \( \ln S_{t} \) in \([0, T]\). It thus follows that \( \ln S_{t} \) has independent increment in \([0, T]\) under any risk-neutral measure.
Finally, we show by induction that for any integer \( n \), any sequence \( \{ T_i \} \in [0, T]^n \) and any real \( U_i \):
\[
Q(S_{T_1} < U_1, \ldots, S_{T_n} < U_n) = P(S_{T_1} < U_1, \ldots, S_{T_n} < U_n).
\]

All risk-neutral probability measures \( Q \) coincide with \( P \) on the algebra \( A \) of cylinders sets \( \{ o \in \Omega : S(T_i, o) < U_1, \ldots, S(T_n, o) < U_n \} \). A direct application of the monotone class theorem as in Jacod and Protter (1991, p 32) implies that these probability measures coincide on \( \sigma(A) \) and thus on \( \mathcal{F}_T \). \( \square \)

We note that the uniqueness of a risk-neutral measure does not, however, imply completeness of the model since the number of traded assets is infinite (see Jarrow and Madan 1999). We have proved that in a sticky-delta implied volatility model, there exists a cadlag square-integrable \( P \)-martingale \( X \) with \( P \)-independent increments such that

\[
S_t = S_0 m_{t_0} \exp(X_t)/E[\exp(X_t)].
\]

Examples of processes with independent increments are Brownian motions, Poisson processes, Levy processes and jump-diffusion processes. For this construction of stochastic integrals with respect to cadlag square-integrable martingales. In this paper, the stochastic integrals from \( a \geq 0 \) to \( b \) are integrals on \( (a, b) \). We recall that the quadratic variation of a cadlag square-integrable martingale \( Z \) is an increasing cadlag adapted process defined by

\[
[Z, Z]_t = Z_t^2 - 2 \int_0^t Z_s^{-} dZ_s.
\]

To ease notation, this process will also be denoted \([Z]_t \) or \([Z]_t \).

Restricting our attention to regular sticky-delta models will ensure that \( X \) has finite moments and satisfy the representation property of Nuallat and Schoutens. This will allow us to prove the quasi-completeness of regular sticky-delta models. But first, we need some notation and some preliminary results.

Since \( S \) is assumed quasi-left continuous, the martingale \( X \) has no fixed points of discontinuity and can be decomposed as in Jacod and Shiryaev (1987, p 77)

\[
X_t = (\sigma \cdot W)_t + (x * (N - n))_t,
\]

so

\[
\sum_{0 < i < T} \Delta X_i = (x * N)_T = \int_0^T \int x N(dx \times ds),
\]

where \( \Delta X_i = X_i - X_{i-} \) is the jump of \( X \) at time \( t \), \( \sigma \) is a square-integrable deterministic function, \( W \) is a Brownian motion, \( N \) is a Poisson measure independent of \( W \) with deterministic compensator \( n \) such that \( n_t(dx) = E[N(dx \times ds)] \) where \( N(dx \times ds) \) is the number of non-zero jumps of \( X \) in \( (s, s + ds) \times (x, x + dx) \).

Since \( \ln S \) and \( X \) are square-integrable, we conclude that for \( i \) in \([0, T]\)
\[
E\left[ \sum_{0 \leq i < s \leq T} \Delta X_i^2 \right] = \int_0^T \int_{-\infty}^{+\infty} x^2 n_u(dx) du < \infty.
\]

We recall the following notation:
\[
(H \cdot W)_t = \int_0^t H_u dW_u,
\]

\[
(\Lambda * (N - n))_t = \int_0^t \Lambda_u \{ N(dx \ du) - n_u(dx) du \}.
\]

By applying Ito's formula (see Jacod and Shiryaev 1987) to \( S_t \), we derive the following representation for the short-dated option prices:

\[
\tilde{C}_{t,s-t,K} = C_{0,0,K} + (H_{0,K}^C \cdot W)_t + (\Pi_{0,K}^C \cdot (N - n))_t \quad (4.6)
\]

\[
\tilde{P}_{t,s-t,K} = P_{0,0,K} + (H_{0,K}^P \cdot W)_t + (\Pi_{0,K}^P \cdot (N - n))_t \quad (4.7)
\]

where \( \tilde{X}_t = P_{0,0,X_t} \). Jensen inequality implies that the martingales \( P_{0,0,C_t(s - t, K)} \) and \( P_{0,0,P_{t,s-t,K}} \) are square-integrable. It follows that \( H^C(s, K), H^P(s, K) \) are in \( L^2_\Pi \) and \( \Pi^C(s, K), \Pi^P(s, K) \) are in \( L^2_\Pi \) with

\[
L^2_\Pi = \{ H \in \Pi : E\left[ \int_0^T H_u^2 du \right] < \infty \}.
\]

\[
L^2_\Pi = \{ \Gamma \in \Pi \otimes B(R) : E\left[ \int_0^T \int \Gamma_u^2(x) n_u(dx) du \right] < \infty \}.
\]

where \( \Pi \) is the predictable \( \sigma \)-algebra on \([0, T] \times \Omega \), that is the smallest \( \sigma \)-algebra making all adapted processes that are left continuous with right limits, measurable (see Protter 1995).

Let \( V_{0, t, K} = P_{0,0,P_{t,s-t,K}} \) and recall that the quadratic variation of this process satisfies with equation (4.7) (see Protter 1995)

\[
[V(t, K), V(t, K)]_t = V_{0, t, K}^2 + \int_0^t H_{u,K}^P(t, K)^2 ds + \int_0^t \int \Gamma_{u,K}^P(t, K)(x)^2 N(dx \times ds).
\]

(4.8)

Let \( \lambda(K) \) be a square-integrable function. The Lebesgue–Fubini theorem for positive integrand (Malliavin and Airault 1994, p 46) implies that the following Lebesgue integrals, if finite, satisfy

\[
\mathbb{E}\left[ \int_0^T \int_{-\infty}^{+\infty} \lambda(K)^2 d[\Gamma_{v,K}^F]_v dK \right]
\]

\[
= \mathbb{E}\left[ \int_0^T \int_{-\infty}^{+\infty} (\lambda(K)^2 H_{v,K}^F)^2 dK \right]
\]

\[
+ \mathbb{E}\left[ \int_0^T \int_{-\infty}^{+\infty} (\lambda(K)^2 \Gamma_{v,K}^P)^2 dK \right] n_u(dx) du.
\]

Since \( S \) and \( 1/S \) are assumed to be square-integrable, we obtain that for all \( i \) in \([0, T] \) and all \( i \geq 0 \)

\[
\mathbb{E}[\ln S_i^4 \exp(\ln S_i)] < i! E[\frac{S_i^2}{S_i^4 - 2}].
\]

(4.9)

It follows that for all \( i \geq 0 \):

\[
\sup_{i \in [0, T]} \{ E[\ln S_i^4 \exp(\ln S_i)] + E[\frac{S_i^2}{S_i^4 - 2}] \} < \infty.
\]

(4.10)

With equation (4.10), we prove the following lemma that will be useful in establishing our quasi-completeness result.
Lemma 4.1. For any integers \( p \geq 0 \) and any real \( \alpha > 0 \), we have
\[
E \left[ \int_0^\alpha \int_0^t (\ln K)^p K^{-\alpha} d\hat{u}_A(t-u, K) dK \right] < \infty, \quad (4.11)
\]
and
\[
E \left[ \int_0^{+\infty} \int_0^t (\ln K)^p K^{-\alpha} d\hat{u}_A(t-u, K) dK \right] < \infty, \quad (4.12)
\]
where \([M] \equiv [M, M], \dot{M} \equiv M/B\) for a cadlag process \(M\).

Proof. Since \( \hat{P}_u(t-u, K) \) is a \( P\)-martingale, we observe that:
\[
E \left[ \int_0^\alpha d\hat{P}_u(t-u, K) \right] = E[\hat{P}_t(0+, K)^2] - \hat{P}_0(t, K)^2.
\]
Therefore, we have
\[
E \left[ \int_0^\alpha \int_0^t (\ln K)^p K^{-\alpha} d\hat{u}_A(t-u, K) dK \right] \leq (\ln K)^p K^{-\alpha} E[(K - S_t)^{2\alpha}].
\]
Using the Lebesgue–Fubini theorem for a positive integrand, we obtain
\[
\int_0^\alpha (\ln K)^p K^{-\alpha} E[(K - S_t)^{2\alpha}] dK \leq E[\ln S_t \mid \mathcal{V}] \ln a^{2\alpha} S_t^{-1}
\]
With (4.10), we conclude that
\[
\int_0^\alpha E \left[ \int_0^t (\ln K)^p K^{-\alpha} d\hat{P}_u(t-u, K) \right] < \infty
\]
By Lebesgue–Fubini, we obtain (4.11). Similarly, we derive inequality (4.12). \(\square\)

Since \(X\) has independent increments and \(E[\exp(2X_t)] + E[\exp(-2X_t)] < \infty\), the Laplace transform of \(X_t\) is defined for all \(|\theta| \leq 2\), \(t \in [0, T]\) and satisfies the time-dependent Levy–Khintchine formula for square-integrable processes:
\[
E[\exp(\theta(X_{t+s} - X_t))] = \exp \left(\int_t^{t+s} \psi_\theta(\lambda) d\lambda\right),
\]
\[
\psi_\theta(\lambda) = \frac{1}{2} \sigma^2 \lambda^2 + \int_{-\infty}^{+\infty} (e^{\theta z} - 1 - \theta z) n_s(dz).
\]
Using the above formula, we show that the martingale \(X\) for a regular sticky-delta model as in definition 2.2 is regular in the following sense.

Definition 4.1. The martingale \(X\) is regular if there exists \(\lambda \geq 2\):
\[
\sup_{x \in (0, T)} \left\{ \sigma^2 + \int_{(-1,1)^T} \exp(\lambda|x|) n_t(dx) \right\} < \infty, \quad (4.13)
\]
Thanks to (4.13), we derive for \(i \geq 2\):
\[
\int_{-\infty}^{+\infty} |x|^i n_s(dx) < \frac{1}{\lambda^i} \int_{(-1,1)^T} \exp(\lambda|x|) n_t(dx) + \int_{-1}^{1} x^2 n_s(dx)
\]
Therefore for \(k, i \geq 2\), \(m_i(t) \equiv \int_{-\infty}^{+\infty} x^i n_s(dx)\) and \(M_k(t) \equiv E\left[\sum_{|x| \leq 1} (\Delta X_t)^i\right] = \int_{-\infty}^{+\infty} m_i(s) ds\) are uniformly bounded in \([0, T]\).

In the case where the increments of \(X\) have stationary distributions, we note that \(X\) is a Levy martingale and the condition (4.13) is similar to the condition introduced by Nualart and Schoutens (2000). With that restriction, we will extend in section 6, the polynomial representation obtained by Nualart and Schoutens for regular Levy martingales, to regular martingales. With this representation property, we will derive our quasi-completeness result in section 7. But first we need to define precisely our admissible trading strategies and what we mean by quasi-completeness, as this type of completeness is not standard.

5. Trading strategies, attainable claims and quasi-completeness

In this section, we define the trading strategies that will be used to replicate contingent claims in a regular sticky-delta model. We first define the static trading strategies, which are static portfolios having a continuum of traded vanilla options. These static trading strategies are traded instruments by assumption (see section 2). As in the Black–Scholes theory, we then define trading strategies as dynamic portfolios involving a finite number of traded instruments.

We highlight that in this paper, static portfolios with a continuum of traded options, i.e. static trading strategies, are assumed to be traded instruments, as in Carr and Madan (1998). We note that this assumption is equivalent to assuming that at time \(t\), all European claims, with maturity \(u \in (t, t + x_m)\) and payoffs \(S_t\), that can be decomposed in a continuum of call and put option payoffs as in Breeden and Litzenberger (1978), are traded instruments.

We define a static trading strategy \(\lambda\) to be a portfolio of short-dated options with maturity \(t_1\) and of the money market account such that the portfolio holdings are constant over the trading interval \((t_0, t_1] \subset (t_0, t_0 + x_m)\) and there are no flows coming in or out of the strategy up to time \(t_1\).

A static strategy is characterized by an initial date \(t_0\), an end date \(t_1 \in (t_0, t_0 + x_m)\), an initial endowment \(C_0 \in \mathcal{I}_{t_0}\), two locally bounded functions \(\lambda^C(K)\) and \(\lambda^F(K)\) defined in \([\alpha, +\infty)\) and in \((0, \alpha)\) respectively, and two positive \(\sigma\)-finite measures \(c_t(dK)\) and \(p_t(dK)\) defined on the positive half line.

The static trading strategy \(\lambda = (t_0, t_1, \lambda^C, \lambda^F, \alpha, c, p)\) is a portfolio that has constant holdings in \((t_0, t_1]\) and that is instantaneously at zero cost using the initial endowment at time \(t_0\) to purchase \(\lambda^C(K)c_t(dK)\) unit(s) of the call option with strike \(K \in (\alpha, +\infty)\) and maturity \(t_1\) and \(\lambda^F(K)p_t(dK)\) unit(s) of the put option with strike \(K \in (0, \alpha)\) and maturity \(t_1\).

We assume that the functions \(\lambda^C(K)\) and \(\lambda^F(K)\) satisfy the following conditions:
\[
E \left[ \int_0^\alpha |\lambda^C_0| \int_{K_0}^K P_{t_0, K} p_t(dK) + \int_\alpha^{+\infty} |\lambda^C_0| \int_{K_0}^K C_0 t K_0 c_t(dK) \right] < \infty, \quad (5.1)
\]
\[
E \left[ \int_0^\alpha \int_0^{t_0} \lambda^F(K)^2 d\hat{P}_{t_0, t_0, K} p_t(dK) \right] < \infty, \quad (5.2)
\]
\[
E \left[ \int_\alpha^{+\infty} \int_0^{t_0} \lambda^C(K)^2 d\hat{P}_{t_0, t_0, K} c_t(dK) \right] < \infty. \quad (5.3)
\]
where we recall \([X] = [X, X]\). Using the Cauchy–Schwarz inequality, we show as in the proof of lemma 4.1, that equations (5.1)–(5.3) are implied by the single equation

\[
E \left[ \int_{S_{t_0}}^{S_{t_1}} \left( \frac{\partial}{\partial a} K \right)^2 p_a(dK) + \int_{a}^{\infty} \left( \frac{\partial}{\partial a} S_a \right)^2 c_a(dK) \right] < \infty.
\]

We impose the holding functions to satisfy (5.2) and (5.3) in order to apply the stochastic Fubini theorem as in Protter (1995, p 160), so as to permute strike and time integrations. As we will see, this will allow us to define dynamic portfolios based on static trading strategies.

The value of the static trading strategy at time \(t \in (t_0, t_1]\) is obtained by adding the values of its constituents:

\[
V_t(\lambda) = \int_{a}^{\infty} \lambda^P(K) p_a(dK) + \int_{a}^{\infty} \lambda^C(K) C_t(1 - t, K)c_a(dK) + C_0 B_t/B_{t_0}.
\]

The above two integrals are guaranteed to exist thanks to (5.1). Since the static portfolio does not involve any cost at initiation, we derive for \(t \in (t_0, t_1]\):

\[
\hat{V}_t(\lambda) = \int_{a}^{\infty} \lambda^P(K) \left( \int_{t_0}^{t} d\hat{P}_t(1 - s, K) \right) p_a(dK) + \int_{a}^{\infty} \lambda^C(K) \left( \int_{t_0}^{t} d\hat{C}_t(1 - s, K) \right) c_a(dK).
\]

Using (5.2) and (5.3) together with (4.6)–(4.9), we show that we can permute strike and time integration, by application of the stochastic Fubini theorem as formulated in Protter (1995) for martingales and in Lebedev (1995) for random measures:

\[
\hat{V}_t(\lambda) = \left[ \left( \int_{a}^{\infty} \lambda^P(K) H_{t_k} p_a(dK) + \int_{a}^{\infty} \lambda^C(K) C_{t_k} c_a(dK) \right) \right] W_{t_0} + \left[ \left( \int_{a}^{\infty} \lambda^P(K) H_{t_k} p_a(dK) + \int_{a}^{\infty} \lambda^C(K) C_{t_k} c_a(dK) \right) \right] (N - n)_{t_0},
\]

where we have used the notation \([X]^a_{t_0} = X_t - X_{t_0}\). In short and with abuse of notations, we simply write

\[
\hat{V}_t(\lambda) = \int_{a}^{\infty} \lambda^P(K) p_a(dK) d\hat{P}_t(1 - s, K) + \int_{a}^{\infty} \lambda^C(K) c_a(dK) d\hat{C}_t(1 - s, K).
\]

We set \(V_t(\lambda) = \hat{V}_t(\lambda|_{\Omega_{0,t_k}, n})\) for \(t \in [0, T]\). This discounted value process is a square-integrable cadlag martingale which is zero before \(t_0\) and constant after \(t_1\).

These static strategies which are portfolios with a continuum of traded options, are traded instruments by assumption.

A trading strategy \(\lambda\) is defined as a portfolio of \(n\) static trading strategies and of the money market account. A trading strategy is thus characterized by a \(\text{finite}\) sequence \(\{\Phi_i : 1 \leq i \leq n\}\) of static trading strategies and by some predictable processes \(\{\lambda_i^\lambda : 1 \leq i \leq n\}\) and by a progressively measurable adapted process \(\lambda^n\) such that

\[
\sup_{t \in (0, T)} E \left[ \left( \sum_{i=1}^{n} \lambda_i^\lambda \hat{V}(\Phi_i) + \lambda^n \right)^2 \right] < \infty. \tag{5.4}
\]

At time \(t\), the strategy \(\lambda\) is the portfolio consisting of \(\lambda_i^\lambda\) unit(s) of each of the static portfolios \(\Phi_i\) and of \(\lambda^n\) unit(s) of the money market account.

The value of such a strategy is given at time \(t\) by the sum of the values of its constituents:

\[
V_t(\lambda) = \sum_{i=1}^{n} \lambda_i^\lambda V_t(\Phi_i) + \lambda^n B_t.
\]

Equation (5.4) guarantees that this process is a square-integrable semimartingale. The gain cumulated up to time \(t\) by the trading strategy is defined as usual by

\[
G_t(\lambda) = \sum_{i=1}^{n} \int_{0}^{t} \lambda_i^\lambda dV_s(\Phi_i) + \int_{0}^{t} \lambda^n dB_s.
\]

Equation (5.5) guarantees the existence of the above integrals. A trading strategy is self-financing if and only if the value process satisfies as usual

\[
V_t(\lambda) = V_0(\lambda) + \sum_{i=1}^{n} \int_{0}^{t} \lambda_i^\lambda d\hat{V}_s(\Phi_i). \tag{5.6}
\]

A self-financing trading strategy is thus entirely characterized by \(\lambda = (\lambda^\lambda, \Phi_i)\) where \(\lambda^\lambda\) is a predictable process satisfying (5.4) and (5.5). We denote by \(\Sigma\) the space of all self-financing trading strategies. We define the following linear subspace of \(L^2(\Omega, \mathcal{F}, \mathbb{P})\), \(V(\Sigma) = \{V_t(\lambda) : t \in [0, T], \lambda \in \Sigma\}\).

As explained in Jarrow and Madan (1999), an arbitrage strategy is a self-financing strategy \(\lambda\) such that

\[
V_0(\lambda) = 0, \quad P(V_T(\lambda) \geq 0) = 1, \quad P(V_T(\lambda) > 0) > 0.
\]

Proposition 5.1. Implied volatility models are arbitrage-free.

Proof. Suppose that \(\lambda\) is an arbitrage strategy then \(E[1|V_T(\lambda) < 0] = 0\) and thus, we have \(E[V_T(\lambda) 1|V_T(\lambda) < 0] = 0\). It follows that \(E[V_T(\lambda)] = E[|\hat{V}_T(\lambda)|] > 0 = V_0(\lambda)\). This contradicts the fact that \(\hat{V}_t(\lambda)\) is a \(P\)-martingale. \(\blacksquare\)

Remark 5.1. This result is not surprising since we have assumed the existence of at least one risk-neutral measure \(P\).

We can now define attainability and attainability up to a vanishing risk.

Definition 5.1. A claim \(\Lambda \in L^2(\Omega, \mathcal{F})\) with maturity \(t \leq T\) is attainable or can be replicated if there is a self-financing trading strategy \(\lambda\) such that \(\Lambda = V_t(\lambda), i.e. \Lambda \in V(\Sigma)\). A claim \(\Lambda \in L^2(\Omega, \mathcal{F})\) with maturity \(t \leq T\) is attainable up to a vanishing risk or can be replicated up to a vanishing risk if there is a sequence of self-financing strategies \(\{\lambda_n\}\) such that \(\lim_{n \to \infty} E[(\Lambda - V_t(\lambda_n))^2] = 0\), i.e. \(\Lambda \in V(\Sigma)\).
A claim $\Lambda \in L^2(\Omega, \mathcal{F}, \mathbb{P})$ with maturity $t$ is thus attainable if there are some static trading strategies $(\vartheta_i = (t_i, \phi_i, \varphi_i, \p_i, c_i, \alpha_i) \geq 0 \leq i \leq n)$, some predictable processes $[\varphi_i]_t \geq 0 \leq i \leq n$ satisfying (5.4) and (5.5) and a real $V_0 = E[P_{t_0}\Lambda]$ which is the cost of replication, such that

$$
P_{t_0}\Lambda = V_0 + \sum_{i=0}^n \int_{t_i}^{t} \int_0^{\infty} \lambda_i^t d\gamma_i(t) dK + \sum_{i=0}^n \int_{t_i}^{t} \int_0^{\infty} \lambda_i^t f_i(t) c_i(t) d\hat{C}_s(t_i - s, K).$$

We say that a claim $\Lambda = \Omega_i(t, b, f) \in L^2(\Omega, \mathcal{F}, \mathbb{P})$ is attainable up to a vanishing risk, i.e. $\lambda \in L^2(\Omega, \mathcal{F}, \mathbb{P})$, such that the claim $\Lambda = \Omega_i(t, b, f)$ is attainable if there are some static trading strategies $P_{t_0}\Lambda = \hat{V}(\lambda) - \hat{V}(\lambda) + C_s$. Similarly, we define attainable up to a vanishing risk, at time $s$.

The third component is trivially replicated by taking position in the money market account. The fourth component is replicated by buying a call option and selling a put option since it is the terminal value of a forward contract.

**Corollary 5.1.** The above result still holds if we replace in (5.7) the squares by absolute values and replicated by replicating up to a vanishing risk.

**Proof.** Consider the sequence of smooth functions $f_n$ defined by replacing in (5.8) $f''(K)$ by $f''(K)1_{|f''(K)| < n}$. By applying the Lebesgue dominated convergence theorem, we obtain $\lim_{n \to \infty} E[(f_n - f)^2] = 0$.

By application of proposition 5.2 to $f_n(S_b)$, we conclude that $f_n(S_b)$ is attainable and thus at time $t$, the claim $f(S_b)$ in $L^2(\Omega, \mathcal{F}, \mathbb{P})$ can be replicated up to a vanishing risk at a cost $P_{t_0}E_t[f(S_b)]$.

**Proposition 5.3.** Let $U_t$ be a square-integrable martingale such that the claim $U_T$ with maturity $T$ is attainable and let $\gamma$ be a predictable bounded process. Then the claim with payoff at time $a$, $Z_a = f_a^{\gamma T} \gamma_t dU_t \in L^2(\Omega, \mathcal{F}, \mathbb{P})$, can be replicated at zero cost.

**Proof.** The claim with payoff $U_T/P_{t_0}\Lambda_T$ is attained by a self-financing trading strategy $\lambda = (\lambda_i, \phi_i)$ with $\hat{V}(\lambda) = E_t[U_T] = U_t$ satisfies

$$U_t = U_0 + \sum_{i=1}^n \int_0^t \lambda_i^t d\hat{V}_t(\phi_i).$$

Since $\gamma$ is a predictable bounded process, $(\gamma_i\lambda_i^t, \phi_i)$ defines a self-financing trading strategy and we have

$$\int_0^t \gamma_t dU_t = \sum_{i=1}^n \int_0^t \gamma_i^t d\hat{V}_t(\phi_i).$$

Therefore, the claim $Z_a$ with maturity $a$ is attainable at zero cost.

We conclude this section with a first illustration of how hedging in a sticky-delta model works in practice (see remark 7.2).

**Proposition 5.4.** All call and put options with maturity in $[0,T]$ are attainable.

**Proof.** Consider the long-dated call option with strike $K > 0$ and maturity $t = kx_m/2 + x$ in $[0,T]$ where $k$ is an integer and $0 < x \leq x_m/2$. Given the set of dates $t = (t_i/2) \cap t : 1 \leq i \leq k + 1$, we shall construct a sequence of static trading strategies $\phi_i$ having trading intervals $[t_i, t_{i+1}]$ such that $(\phi_i)$ replicates the long-dated call option. At time $t_k = kx_m/2$, the call option is a short-dated option and thus can be replicated by purchasing the call option itself. The discounted value process associated with this static
strategy is $\hat{V}_d = \hat{C}_d(t-u,K)$ for $u \in [t_k, t]$. Given the sticky-delta assumption, we obtain

$$\hat{V}_{t_k} = P_{t_k}E[(S_t - K)^+ | S_{t_k}] = P_{t_k}S_{t_k}c(t-t_k, K/S_{t_k}),$$

where $c(x, y) = \mathbb{E}[(S_t / S_{t-k} - y)^+]$ is a $C^2$ function which is decreasing and convex with respect to the second argument. The density of $S_t / S_{t-k}$ is $\partial_y^2 c(x, y)$ and $\partial_y^2 c(x, y)$ is uniformly bounded in $(0, +\infty)$. We note that $\hat{V} \in L^2(\Omega, \mathcal{F}, \mathbb{P})$ and $\partial_y^2 \hat{V}_{t_k} = K^2 \partial_y^2 c(\Delta t_k, K/S_{t_k}) / S^2_{t_k} \geq 0$. We derive

$$E \left[ \int_{S_{t_k}}^{1} \partial_y^2 c(\Delta t_k, K/S_{t_k}) S^2_{t_k} / S^4_{t_k} dK \right] < \infty,$$

$$E \left[ \int_{S_{t_k}}^{1} \partial_y^2 c(\Delta t_k, K/S_{t_k}) S^2_{t_k} / S^4_{t_k} dK \right] < \infty.$$

We can thus apply proposition 5.2 and replicate the claim $\hat{V}_{t_k}$ between time $t_{k-1}$ and $t_k$ using a static portfolio with positive holdings in short-dated call and put options having maturity $t_k$, at a discounted cost

$$\hat{V}_{t_{k-1}} = P_{t_{k-1}}E[(S_t - K)^+ | S_{t_{k-1}}] = P_{t_{k-1}}S_{t_{k-1}}c(t-t_{k-1}, K/S_{t_{k-1}}).$$

By repeating the above argument, we obtain a set of static trading strategy $\phi_i : 0 \leq i \leq k$ with positive holdings such that the self-financing trading strategy $(\phi_i)$ replicates the long-dated call option at the following initial cost:

$$\hat{V}_0 = P_0 E[(S_t - K)^+] + \sum_{i=1}^{k} \int_0^T \phi_i(t) dH_i(t),$$

where $\phi_i(t) = \mathbb{E}[(S_t - K)^+]$ is such that the martingales $H_i(t)$ are square-integrable and pairwise strongly orthogonal.

In order to construct similar hedging strategies for square-integrable claims, we need a representation property for $X$.

### 6. Representation property for regular martingales with independent increments

In this section and in the appendix, we extend the representation property obtained by Nualart and Schoutens (2000) for Levy processes, to regular martingales with independent increments. We adapt the notation and the approach taken by Nualart and Schoutens to our purpose.

We define the Teugels martingales on $[0, T]$ as in Nualart and Schoutens (2000):

$$Y^{(i)}_t = X_t,$$

$$Y^{(k)}_t = \sum_{0 \leq s \leq t} (\Delta X_s)^k - \int_0^t m_k(s) ds, \hspace{1cm} (k \geq 2).$$

It is clear that these martingales are square-integrable with independent increments and finite moments of all orders. In the appendix and by $[]$-orthogonalization of the Teugels martingales, we construct the following family of pairwise $[]$-orthogonal martingales having independent increments:

$$H^{(i)}_t = Y^{(i)}_t.$$
Proposition 7.1. For $i \geq 1$ and $b \in (a, a + x_m)$, $H_b^{(i)} - H_u^{(i)}$ satisfies

$$H_b^{(i)} - H_u^{(i)} = \int_a^b \int_0^{s_n} h_u^{(i)}(K)K^{-2} dK d\hat{P}_u(b - u, K)$$

$$+ \int_a^b \int_0^{s_n} g_u^{(i)}(K)K^{-2} dK d\hat{C}_u(b - u, K)$$

$$+ \int_a^b f_u^{(i)}(K)K^{-2} dK d\hat{P}_u(b - u, f_0b)$$

$$h_u^{(i)}(K) = \sum_{0 \leq j \leq k < i - 1} h_{u,j}^k X_{u,j}^k (\ln(K/f_0b))^k,$$

$$g_u^{(i)}(K) = \sum_{0 \leq j \leq k < i - 1} g_{u,j}^k X_{u,j}^k .$$

The coefficients $h_{u,j}^k$ and $g_{u,j}^k$ are deterministic and uniformly bounded on $[0, T]$. For all $i \geq 1$, the claim $H_T^{(i)}$ is attainable, i.e. $H_T^{(i)} \in V(\Sigma)$.

Proof. $Y_t^{(i)} = \ln(S_t/f_0b)$ is a square-integrable martingale. By proposition 5.2, we conclude that $Y_t^{(i)}$ is attainable using the static trading strategy defined by

$$P_{0b}Y_t^{(i)} = -\int_a^b \int_0^{s_n} \hat{P}_t(b - u, K)K^{-2} dK$$

$$- \int_a^b \int_0^{s_n} \hat{C}_t(b - u, K)K^{-2} dK$$

$$+ (\hat{C}_t(b - u, f_0b) - \hat{P}_t(b - u, f_0b))/f_0b .$$

This equation implies for $t \in [a, b]$

$$Y_t^{(i)} - Y_a^{(i)} = \int_a^t \int_0^{s_n} y_u^{(1)}(K)K^{-2} dK d\hat{P}_u(b - u, K)$$

$$+ \int_a^t \int_0^{s_n} y_u^{(2)}(K)K^{-2} dK d\hat{C}_u(b - u, K)$$

$$+ \int_a^t \int_0^{s_n} y_u^{(3)}(K)K^{-2} dK d\hat{P}_u(b - u, f_0b)$$

where $y_u^{(i)}(K) = -\frac{1}{\sigma_b^2}$ and $y_f^{(i)} = \frac{1}{f_0b}$.

With Ito’s formula for a real function $f$ of class $C^2$,

$$\sum_{0 \leq a \leq t} \Delta f(X_a) - f(X_{a-})\Delta X_a = f(X_t) - f(X_0)$$

$$- \int_0^t f'(X_{a-}) dX_u - \frac{1}{2} \int_0^t f''(X_{a-}) \sigma_b^2 du .$$

Taking $f(x) = x^2$, we derive:

$$Y_b^{(2)} = E[Y_b^{(2)}] - E[f(X_b^{(2)})] - 2 \int_0^b X_u - dY_u^{(3)} .$$

Finally, the above equation and lemma 7.1 imply that $Y_b^{(2)} - Y_a^{(2)}$ is attainable:

$$Y_b^{(2)} - Y_a^{(2)} = \int_a^b \int_0^{s_n} y_u^{(1)}(K)K^{-2} dK d\hat{P}_u(b - u, K)$$

$$+ \int_a^b \int_0^{s_n} y_u^{(2)}(K)K^{-2} dK d\hat{C}_u(b - u, K)$$

$$+ \int_a^b f_u^{(1)}(K)K^{-2} dK d\hat{P}_u(b - u, f_0b)$$

$$y_u^{(2)}(K) = 2[X_u - \ln(K/f_0b) + 1]/P_{0b}$$

$$f_u^{(2)} = -2X_u - f_u^{(1)} .$$

Similarly, we obtain by applying Ito’s lemma to $f(x) = x^3$:

$$\sum_{0 < u < b} (\Delta X_u)^3 - \int_0^b m_3(s) ds = (X_b)^3 - E[(X_b)^3]$$

$$- 3 \int_0^b X_u - dY_u^{(3)} + 3 \int_0^b (X_u - V_b - V_a + M_u^{(2)} - M_u^{(3)}) dy_u^{(3)}$$

where $V_t = \int_0^t \sigma^2 ds$ and $M_t^{(2)} = \int_0^t m_2(s) ds$.

It follows that:

$$Y_b^{(3)} - Y_a^{(3)} = \int_a^b \int_0^{s_n} y_u^{(1)}(K)K^{-2} dK d\hat{P}_u(b - u, K)$$

$$+ \int_a^b \int_0^{s_n} y_u^{(2)}(K)K^{-2} dK d\hat{C}_u(b - u, K)$$

$$+ \int_a^b f_u^{(1)}(K)K^{-2} dK d\hat{P}_u(b - u, f_0b)$$

$$y_u^{(3)}(K) = Q_u^{(2)}(X_u - \ln(K/f_0b))$$

$$f_u^{(3)} = R_u^{(2)}(X_u)$$

and the polynomials $Q_u^{(2)}(X, Y) = \sum_{0 \leq i \leq k < i - 1} q_{u,i}^{(2)} X^i Y^k$ and $R_u^{(2)}(X) = \sum_{0 \leq i \leq k < i - 1} r_{u,i}^{(2)} X^i$ have coefficients that are deterministic and uniformly bounded in $[0, T]$.

By induction, we extend the above formula to all $Y_b^{(k)} - Y_a^{(k)}$. We recall that:

$$[H_b^{(k)}]_{0b} = \int_a^b dY_u^{(k)} + a_{2,k-1}(u) dY_u^{(k-1)} + \ldots + a_{1,k}(u) dY_u^{(1)},$$

where the coefficients are deterministic and bounded in $[0, T]$.

We finally obtain the promised expression for $H_b^{(k)} - H_a^{(k)}$ by using the previous equation for $Y_b^{(k)} - Y_a^{(k)}$. By adding the above decompositions obtained for $a = t_{l-1}, b = t_l$ with $l = 1, \ldots, [2T/x_m] + 1, t_l = (lx_m/2) \wedge T$, we obtain a self-financing trading strategy that replicates $H_T^{(k)}$.

Finally, we have the following quasi-completeness result.

Theorem 7.1. Any square-integrable claim $\Lambda \in L^2(\Omega, \Sigma)$ with maturity $T$ can be replicated up to a vanishing risk at a cost $P_{0T} E[\Lambda]$ by trading the underlying, the money-market account and some portfolios of traded call and put options. A regular sticky-delta model is quasi-complete or complete up to a vanishing risk.

Proof. According to proposition 6.1, the random variable $\Lambda \in L^2(\Omega, \Sigma)$ can be represented as follows:

$$\Lambda = E[\Lambda] + \sum_{i=1}^{\infty} \int_0^T \xi_i^{(i)} dH_i^{(i)} ,$$

where $\xi_i^{(i)}$ is in $L^2_F^{(i)}$.

We define $\Lambda_i = E[\Lambda] + \sum_{i=1}^{n} \int_0^T \xi_i^{(i)} dH_i^{(i)} \in V(\Sigma)$ with $\xi_i^{(i)} = \xi_i^{(i)}(1[|\xi_i^{(i)}| < n]}$.
Thanks to the strong orthogonality of $H_{i}^{(2)}$, we obtain
\[
E[(\Lambda - \Lambda_n)^2] \leq E \left[ \left( \Lambda - E[\Lambda] - \sum_{i=1}^{n} \int_{0}^{T} \xi^{(i)}_{s} dH_{s}^{(i)} \right)^2 \right] + E \left[ \sum_{i=1}^{n} \int_{0}^{T} (\xi^{(i)}_{s} - \xi^{(i,n)}_{s})^2 d[H^{(i)}, H^{(i)}] \right].
\]

The first sequence on the RHS converges to zero since $\{\xi^{(i)}_{s} : i \geq 1\}$ belongs to $\mathcal{E}_{\mathcal{L}^{2}} \mathcal{L}^{2}$. Observe next that:
\[
\lim_{n \to \infty} (\xi^{(i,n)}_{s} - \xi^{(i)}_{s})^2 = 0, \quad (\xi^{(i,n)}_{s} - \xi^{(i)}_{s})^2 \leq 4(\xi^{(i)}_{s})^2.
\]

By three applications of the dominated convergence Lebesgue theorem, we derive
\[
\lim_{n \to \infty} E \left[ \sum_{i=1}^{n} \int_{0}^{T} (\xi^{(i)}_{s} - \xi^{(i,n)}_{s})^2 d[H^{(i)}, H^{(i)}] \right] = 0.
\]

Finally, we have $\lim_{n \to \infty} E[(\Lambda - \Lambda_n)^2] = 0$. Each claim $\Lambda_n$ is attainable by application of propositions 7.1 and 5.3 at a cost $P_{0T} E[\Lambda]$. □

**Remark 7.1.** The concept of quasi-completeness is due to Jarrow and Madan (1999) and to Bjork et al (1997).

The next proposition shows that some square-integrable claims can be replicated in the classical sense, by rolling a portfolio of vanilla options.

**Proposition 7.2.** Consider a claim with square-integrable payoff $f(S_{T_1}, \ldots, S_{T_n})$ at time $T_n$. We assume that this payoff is such that $S_{T_i} \mapsto E[f(S_{T_1}, \ldots, S_{T_n})]$ satisfies, after subtraction of a finite number of call and put payoffs, equation (5.7) with $b = T_1$. Then the claim can be replicated by trading the underlying, the money-market account and some portfolios of traded call and put options.

**Proof.** Consider a claim with maturity $T_n$ such as an Asian option, a discrete barrier option, a Parisian option or a volatility swap, that has a finite number of fixing dates $T_i$ with $0 < T_i - T_{i-1} < \infty$ and a square-integrable payoff $f(S_{T_1}, \ldots, S_{T_n})$ as in proposition 7.2. At time $T_{n-1}$, the payoff can be replicated using a portfolio of call and put options, a forward contract and a zero coupon bond as in proposition 5.2. The value $V_{T_{n-1}}$ of the complex option at time $T_{n-1}$, is thus
\[
V_{T_{n-1}} = P_{T_{n-1}, \Delta T_{n-1}} \times [\varphi_{T_{n-1}, \Delta T_{n-1}, f}(S_{T_1}, \ldots, S_{T_{n-1}}, \bullet)](S_{T_{n-1}}),
\]
where the linear operator $\varphi_{T_{n-1}}$ is defined by
\[
[\varphi_{T_{n-1}} f](S) = \int_{0}^{+\infty} f(I \times S_{T_{n-1}}) dI.\]

The sticky-delta assumption implies that the operator $\varphi_{T_{n-1}, \Delta T_{n-1}}$ is deterministic and thus $V_{T_{n-1}}$ is a deterministic function of $S_{T_1}, \ldots, S_{T_{n-1}}$. At time $T_{n-2}$, the complex option can thus be replicated by the use of calls, puts, forwards and zero coupon bonds with maturity $T_{n-1}$. Finally, we derive by induction that the complex option can be replicated at a cost
\[
V_0 = P_{0T} \left[ \varphi_{0,T} \prod_{i=0}^{n-1} \varphi_{T_{i}, \Delta T_{i}} \right](f)(S_0) = P_{0T} E[f].
\]

**Remark 7.2.** By modifying the above strategy, we obtain a super-replication strategy for the complex option when there are transaction costs on the implied volatility or when the short-dated implied volatility smile is ‘uncertain’ but bounded. Observe that the transition operators $\varphi_{T_{n}, \Delta T_{n}}$ are nonlinear in this case.

The previous expectation can be estimated by using the Monte Carlo method, a fast Fourier transform as in Carr and Madan (1999) or by solving the integro-differential equation $\partial_t V + A_t V = r_t V$ where $A_t$ is the generator defined by $A_t = (\delta_{\text{I}} \varphi_{\text{I}})_{t=0}^T$.

**8. Regular geometric Levy models**

We consider a regular sticky-delta implied volatility model and we suppose that the per-delta implied volatility processes are independent of time. Our previous analysis in section 4 shows that the increments of $\ln S_t$ are independent and have a stationary distribution under the probability measure $P$, i.e.
\[
S_t = F_0 \exp (L_t - \ln E[e^{L_t}]),
\]
where $L$ is a regular $P$-Levy martingale (see Levy 1965).

We define a regular geometric Levy under $P$ by:
\[
S_t = F_0 \exp (L_t - \ln E[e^{L_t}]),
\]
\[
C_t(x, K) = P_{tx} E_t [(S_{tx} - K)^+]^+, \quad P_{tx} E_t [(K - S_{tx})]^+,
\]
where $x < x_0$ and $L$ is a local martingale.

**Proposition 8.1.** Regular geometric Levy models are arbitrage-free and quasi-complete in the sense that all square-integrable claims can be replicated up to a vanishing risk by trading the underlying, the money-market account and portfolios of short-dated call and put options.

**Proof.** By direct calculation, we show that regular geometric Levy models are regular stationary sticky-delta implied volatility models. Therefore, these models are arbitrage-free and quasi-complete by application of theorem 7.1. □

9. Characterization of sticky-strike implied volatility models

As with sticky-delta models, we consider only regular sticky-strike models. We denote $v_{xK}(K) \equiv x \Sigma_i(x, Km_{xK})^2$, the density of $S_t$ by $p_t$ and $\Gamma_{1K}(S, K) \equiv \Gamma_{BS}(x \Sigma_i(x, Km_{xK})^2, S, K)$.

**Lemma 9.1.** For each $t < T$, there exists a positive, locally bounded, $P$-integrable function $h_t$ such that:

$$\lim_{k \to +\infty} \partial_x v_{tx}(S_t) = h_t(S_t) \quad P \text{ a.s.,}$$

where $x_k$ is a sequence with limit zero.

**Proof.** First, we note by Jensen inequality:

$$E_t[(S_{tx}/m_{tx} - K)^+] = E_t[(S_{tx+y}/m_{tx+y} - K)^+]$$

Hence $v_{tx}(K)$ is increasing with $x$. To simplify notation, we assume that $p_t > 0$ in $(0, +\infty)$. The case where $p_t$ vanishes is treated similarly since we have $P(p_t(S_t) = 0) = 0$.

Using the Black–Scholes equation, we derive:

$$\partial_t E_t[S_{tx}^2/F_{tx}^2] = \int_0^\infty \partial_x v_{tx}(K) \Gamma_{tx}(S_t, K) dK.$$  

For $I \subset (0, +\infty)$, we define the increasing function:

$$H_{tx}: z \mapsto \int_I \partial_x v_{tx}(K) E_t[\Gamma_{tx}(S_t, K)] \times 1_{\{S_t < z\}}dK.$$  

We observe that for any positive $z$:

$$H_{tx}(z) \leq E_t[\sup \partial_x E_t[S_{tx}^2/F_{tx}^2]].$$  

Definition 2.2 implies that the increasing functions $H_{tx}$ are uniformly bounded with respect to $I$ and $x$. Helly’s theorem implies that there is a sequence $x_{I,n}$ with limit zero and an increasing function $H_{tx}$ such that (see Doob 1994):

$$\lim_{n \to +\infty} H_{tx}(z) = H_{tx}(z).$$

The increasing function $H_{tx}$ satisfies for $0 \leq a < b$:

$$H_{tx}(b) - H_{tx}(a) \leq \int_a^b \sup_x \partial_x E_t[S_{tx}^2/F_{tx}^2] S_t dS_t.$$  

The Radon–Nikodym theorem implies that there is a positive, locally bounded, $P$-integrable function $h_{tx}$ such that:

$$H_{tx}(b) - H_{tx}(a) = \int_a^b h_{tx}(z) p_t(z) dz \quad (a, b > 0).$$  

Finally, we conclude that for any compact $A \subset (0, +\infty)$:

$$\lim_{n \to +\infty} \int_I \partial_x v_{txI,n}(S_{txI,n}(S_t, K) 1_A(S_t)) dK = \int_A h_{tx} p_t dK.$$  

Since $E[\Gamma_{txI}(S_{txI})/p_t]$ converges uniformly to $1_A$ on $I$:

$$\lim_{n \to +\infty} \int_I \partial_x v_{txI,n} 1_A(K) p_t(K) dK = \int_A h_{tx} p_t dK.$$  

Hence $\partial_x v_{txI}(S_t)$ converges in probability to $h_{tx}(S_t)$ in $I$ and there exists $x_{I,t}(n)$ with limit zero, such that (see Doob 1994):

$$\lim_{n \to +\infty} \partial_x v_{txI,n}(S_t) = h_{tx}(S_t) \quad (S_t \in I, P \text{ a.s.}).$$  

Application of Fatou’s theorem gives (see Doob 1994):

$$\lim_{n \to +\infty} \int_I E[\Gamma_{txI,n}(S_t, K) 1_A(S_t)] = [\partial_x v_{txI,n} - h_{tx}] dK = 0.$$  

We define $I_k = \left(\frac{1}{k+1}, \frac{1}{k}\right) \cup [k, k + 1)$ and construct, as previously, a family of positive $P$-integrable functions $h_{I_k}$ and sub-sequences $x_{I_k(n)} \subset x_{I_k(n)}$ converging to zero such that:

$$\lim_{n \to +\infty} \partial_x v_{txI_k(n)}(S_t) = h_{I_k}(S_t) \quad (k > 0, S_t \in I_k, P \text{ a.s.})$$  

Using the diagonal procedure, we define the sequence $x_{\phi(n)} = x_{I_k(n)}$ with limit zero, and the positive, locally bounded, $P$-integrable function:

$$h_t(S) = \sum_{k=1}^{+\infty} 1_I(h_{I_k}(S)) \leq \sup_x \partial_x E_t[S_{tx}^2/F_{tx}^2] S_t = S.$$  

Using equation (9.1), we obtain for any compact sets $C, A$:

$$\lim_{n \to +\infty} \int_C E[\Gamma_{txI_k(n)}(S_t, K) 1_A(S_t)] \partial_x v_{txI_k(n)} - h_{I_k} dK = 0.$$  

Since $\bigcup_k I_k = (0, +\infty)$, we finally conclude that:

$$\lim_{n \to +\infty} \partial_x v_{txI_k(n)}(S_t) = h_t(S_t) \quad P \text{ a.s.}$$  

□

In fact, we have a stronger result.

**Theorem 9.1.** In a regular sticky-strike implied volatility model, the per-strike implied volatility $\Sigma_t(x, K)$ is independent of $K$. Hence the Black–Scholes model is the only arbitrage-free regular sticky-strike model.

**Proof.** The implied volatility model with zero-drift underlying $S_t/m_0$ and implied volatility $\Sigma_t(x, Km_0)$ is regular and sticky-strike. Therefore, there is no loss of generality in assuming zero drift i.e. $m_0 = 1$.

For any bounded, Borel measurable function $f$, we have:

$$P_t E_t [f(S_{tx})] = \int_0^{+\infty} \partial_K C_t(x, K) f(K) dK.$$  

With our assumptions, the asset price process $S_t$ is a Markov process entirely characterized by the transition function $\Phi_{tx}$ (see Revuz and Yor 1991):

$$(\Phi_{tx} f)(S_t) = E[f(S_{tx})|S_t].$$  

41
The transition function is Feller and it is thus associated with
an infinitesimal generator $A_t$ defined on $D_A$ satisfying:

$$[\tilde{A}_t, f](S_t) = \lim_{x \to 0} \partial_x [\tilde{v}_{\varepsilon tx}(f)(S_t)].$$

Let $f \in D_A \cap C^2$ with compact support $\Theta$. We obtain:

$$\partial_x [\tilde{v}_{\varepsilon tx}(f)](S_t) = \int_\Theta \partial_x C^{BS}(v_{\varepsilon tx}(K), S_t, K) f''(K) dK. \quad (9.3)$$

The Black–Scholes function satisfies:

$$\partial_v C^{BS}(V, F, K) = \frac{1}{2} F^2 \Gamma^{BS}(V, F, K).$$

Hence, equation (9.3) can be written as follows:

$$\partial_t [\tilde{v}_{\varepsilon tx}(f)](S_t) = \frac{1}{2} \int_\Theta S_t^2 \Gamma^{BS}(S_t, K) \times \partial_x v_{\varepsilon tx}(f''(K) dK.$$

According to lemma 9.1, there is a positive, locally bounded $P$-integrable function $h_t$ such that:

$$\lim_{n \to \infty} \partial_x v_{\varepsilon tx}(S_t) = h_t(S_t) \quad P \text{ a.s.}$$

By taking first the expectation of equation (9.3) on a compact $C$ and then the limit as $x_\varepsilon$ tends to zero, we obtain with (9.2):

$$\int_C [\tilde{A}_t, f](K) p_t(K) dK = \frac{1}{2} \int_C h_t(K) K^2 f''(K) p_t(K) dK.$$

We finally obtain the following expression for the generator:

$$[\tilde{A}_t, f](S_t) = \frac{1}{2} h_t(S_t) S_t^2 f''(S_t) \quad P \text{ a.s.} \quad (9.4)$$

The forward price $V(t, S_t)$ of a call option with maturity $a$ and strike $K$ is a $P$-martingale. Therefore, $V$ satisfies the backward equation (see Revuz and Yor 1991):

$$\partial_t V(t, S_t) + [\tilde{A}_t, V](S_t) = 0. \quad (9.5)$$

On the other hand, $V$ satisfies:

$$V(t, S_t) = C^{BS}((a - t) \Sigma_t(a - t, K)^2, S_t, K).$$

Since $V$ has the same second-order derivatives with respect to $S$ as the Black–Scholes function, it follows that:

$$\partial^2 V(t, S_t) = \frac{1}{2} \partial^2_t [((a - t) \Sigma_t(a - t, K)^2] S_t^2 \partial^2 S V(t, S_t). \quad (9.6)$$

Finally, equations (9.4)–(9.6) imply that we have for all $K, t, a \in (t, t + x_m)$:

$$\Sigma_t(a - t, K)^2 = \frac{1}{a - t} \int_{a - t}^a E[h_s(S_o)] dT.$$

Hence, the implied volatility $\Sigma_t(x, K)$ is independent of $K$ and the regular sticky-strike model coincides with the Black–Scholes model! \hfill \Box

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Appendix. Representation property for regular martingales
In this appendix, we extend the Schoutens–Nualart representation property obtained for regular Levy martingales to regular martingales with independent increments. We use the notations of section 6 and we follow closely the presentation of Nualart and Schoutens (2000).

We recall that two square-integrable martingales $M$ and $N$ are said to be strongly orthogonal or $\langle \cdot \rangle$-orthogonal if $[M, N]_t$ is a martingale (see Protter 1995).

**Proposition A.1.** There exists a family of pairwise strongly orthogonal square-integrable martingales $[H^{(i)} : i \geq 1]$.

**Proof.** We define the following inner product acting on the space of real polynomials with time-dependent coefficients in $L^2(0, \infty)$:

$$\langle Q, R \rangle = \int_0^T \int_{-\infty}^\infty Q(x, s) R(x, s) (x^2 n_t(dx) - \sigma^2_t \delta(x) dx) ds.$$

By $\langle \cdot \rangle$-orthogonalization of the total family $\{i \delta < t \mid x^i : i \geq 0, t \in (0, T]\}$, we find a family of pairwise $\langle \cdot \rangle$-orthogonal polynomials $\{R_i(x, s)\}$ with bounded coefficients such that:

$$R_i(x, s) = a_{i+1,1}(s) + a_{i+1,2}(s)x + \ldots + a_{i+1,i}(s)x^{i-1} + x^i,$$

$$\langle R_i, \{i \delta < t \mid x^i \rangle = 0, \quad (t < T, 1 \leq i \leq i - 1).$$

The coefficients of the above polynomial are bounded because the measure $\mu(s, dx)$ has finite moments of order $i \geq 2$, uniformly bounded in $[0, T]$.

Define $Q_{i+1}(x, s) \equiv x R_i(x, s) - a_{i+1,1}(s)$, $a_{i+1,i+1} \equiv 1$ and the following square integrable martingale:

$$H^{(i+1)}_t \equiv \int_0^t a_{i+1,1}(s) dY^{(i+1)}_s + \ldots + a_{i+1,i+1}(s) dY^{(i+1)}_t.$$

By direct calculation, we derive:

$$H^{(i+1)}_t = \int_0^t a_{i+1,1}(s) dX_s + \sum_{0 \leq s < t} Q_{i+1}(\Delta S_s, s)$$

$$- \sum_{k=0}^{i+1} \int_0^t a_{i+1,k}(s) m_k(s) ds.$$
Therefore $H^{(i+1)}$ is strongly orthogonal to all $Y^{(j)}$ for $j = 1, \ldots, i$. The martingales $H^{(i)}$ are consequently pairwise strongly orthogonal.

**Remark A.1.** We observe that if the polynomial $R_i$ is such that $\langle R_i, R_j \rangle = 0$ then the martingale $H^{(i+1)}$ satisfies $E[H^{(i+1)}] = 0$ and thus $H^{(i+1)} = 0$ almost everywhere. If the discontinuous component of the regular martingale is the sum of a finite number of Poisson processes with deterministic intensity then only a finite number of the martingales $H^{(k)}$ is non-zero.

We define the following families of $L^2(\Omega, \mathcal{G})$ variables:

$$
\mathcal{E}_{t_i} = \{X_{i_1}^{(1)}, \ldots, X_{i_k}^{(k)} : k \geq 0\},
$$

$$
\mathcal{E} = \{X_{i_1}^{(1)}, \ldots, X_{i_k}^{(k)} : 0 < i_1 < \cdots < i_k < \infty, k \geq 0\}.
$$

**Lemma A.2.** The family $\mathcal{E}_{t_i}$ is total in the space $L^2(\Omega, \sigma(X_{t_1}, X_{t_2}, \ldots, X_{t_k}, X_{t_k+1}, \ldots, X_{t_k+2})).$

**Proof.** Let us prove the result for the family $\mathcal{E}_t$. The case with $n$ non-overlapping independent increments is treated similarly. Since $X$ is regular, the linear hull $\mathcal{E}_t$ of $\mathcal{E}$ is in $L^2(P(X_t \in dx)).$ The space of functions in $L^2(P(X_t \in dx))$ having compact support is dense in $L^2(P(X_t \in dx)).$ Let $F \in C$ with support in $[-a, a]$ and weakly orthogonal to $\mathcal{E}_t$, i.e. $E[(X_s)^2F(X_t)] = 0,$ $(k \geq 0).$ For any real $Z$, we have:

$$
\sum_{k=0}^{+\infty} |Z|^k k! E[(X_s)^kF(X_t)] < E(F(X_s)^2)^{1/2} \exp(|Z| a).
$$

By the dominated convergence theorem, we derive $E[\exp(iZX_s)F(X_t)] = 0$ for all $Z$ and thus $F(X_t) = 0$ (Malliavin and Airault (1994, p 110)). Therefore $\mathcal{E}_t$ is dense in $C$ and thus in $L^2(P(X_t \in dx)).$

**Proposition A.2.** The family $\mathcal{E}$ is total in $L^2(\Omega, \mathcal{G})$.

**Proof.** A variable $Z$ in $L^2(\Omega, \mathcal{G})$ can be approximated arbitrarily closely by an element $Y$ of $L^2(\Omega, \sigma(X_{t_1}, X_{t_2}, \ldots, X_{t_k}, X_{t_k+1}, \ldots, X_{t_k+2}))$ for some sequence $\{t_i\}$. Following lemma A.2 $\mathcal{E}_{t_i}$ is total in $L^2(\Omega, \sigma(X_{t_1}, \ldots, X_{t_k})).$. Therefore, $Y$ and thus $Z$ can be approximated arbitrarily closely by an element of $\mathcal{E}.$

We have the following representation property for elements of $\mathcal{E}$.

**Lemma A.3.** For any integer $k$ and any power-increment $(X_{s_k} - X_{s_k})^k$, there is a sequence of predictable processes $[\theta^{(i)}_{s_k}] = \{s_k : i \geq 1\}$ such that:

$$
(X_{s_k} - X_{s_k})^k = E[(X_{s_k} - X_{s_k})^k] + \sum_{i=1}^{+\infty} \int_{s_k}^{s_k+1} \theta^{(i)}_{s_k}(s) \, dH^{(i)}.
$$

**Proof.** We prove the above equation by induction on $k$ and by application of Ito’s formula to power functions as in Nualart and Schoutens (2000).

**Proposition A.3.** Let $R \in \mathbb{E}$; then there are predictable processes $\{\Psi^{(i)}_t : i \geq 1\}$ such that:

$$
R = E[R] + \sum_{i=1}^{+\infty} \int_0^T \Psi^{(i)}_t \, dH^{(i)}.
$$

**Proof.** With lemma A.3, we derive that the product of non-overlapping power-increments $Y_{t_i} = \{X{^{(i)}}_t \}X{^{(i)}}_t$ with $s_0 < s_1 < s_2 < s_3$ can be represented as follows:

$$
Y_{t_i} = E[Y_{t_i}] + \sum_{i=1}^{+\infty} \int_0^T \Psi^{(i)}_t \, dH^{(i)}.
$$

The process $\Psi^{(i)}_t$ is a predictable process defined by:

$$
\Psi^{(i)}_t = \{s \in (s_2, s_3) \} \theta^{(i)}(s) \sum_{j=1}^{+\infty} \int_{s_0}^{s_1} \theta_{s_0, s}(u) \, dH^{(i)}.
$$

Hence, we have proved the result for the product of two non-overlapping power-increments. By induction, we prove the result for the product of an arbitrary number of non-overlapping power-increments.

Since the linear space $\mathcal{E}$ spanned by $\mathcal{E}$ is dense in $L^2(\Omega, \mathcal{G})$, we deduce the following representation property of square-integrable variables.

**Proposition 6.1.** Let $F \in L^2(\Omega, \mathcal{G})$ then there is a family of predictable processes $\{\phi^{(i)} : i \geq 1\}$ such that:

$$
F = E[F] + \sum_{i=1}^{+\infty} \int_0^T \phi^{(i)} \, dH^{(i)}.
$$

where $\phi^{(i)}(t)$ belongs to $L^2_{H^{(i)}}$.

**Proof.** For a square-integrable adapted martingale $m$, we recall that:

$$
L^2_{H^m} = \left\{ H \in \Pi : \int_0^T H_t \, dm_t \in L^2(\Omega, \mathcal{G}) \right\},
$$

where $\Pi$ is the predictable $\sigma$-algebra on $[0, T] \times \Omega$. We need to prove that:

$$
L^2(\Omega, \mathcal{G}) = \bigoplus_{i=1}^{+\infty} L^2_{H^{(i)}},
$$

$$
\bigoplus_{i=1}^{+\infty} L^2_{H^{(i)}} = \left\{ X \in L^2(\Omega, \mathcal{G}) : X = \sum_{i=1}^{+\infty} \phi^{(i)} \otimes H^{(i)}, \phi^{(i)} \in L^2_{H^{(i)}} \right\}.
$$

We observe that $\bigoplus_{i=1}^{+\infty} L^2_{H^{(i)}}$ is closed in $L^2(\Omega, \mathcal{G})$ since the martingales $H^{(i)}$ are pairwise strongly orthogonal. Thanks to proposition A.3, we have:

$$
\mathcal{E} \subset \bigoplus_{i=1}^{+\infty} L^2_{H^{(i)}} \subset L^2(\Omega, \mathcal{G}).
$$

Since $\mathcal{E} = L^2(\Omega, \mathcal{G})$ by proposition A.2, we derive the result by closure of the above inclusion.
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