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Variance reduction for Monte Carlo simulation in a stochastic volatility environment

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Abstract

We propose a variance reduction method for Monte Carlo computation of option prices in the context of stochastic volatility. This method is based on importance sampling using an approximation of the option price obtained by a fast mean-reversion expansion introduced in Fouque *et al* (2000 *Derivatives in Financial Markets with Stochastic Volatility* (Cambridge: Cambridge University Press)). We compare this with the small noise expansion method proposed in Fournie *et al* (1997 *Asymptotic Anal.* **14** 361–76) and demonstrate numerically the efficiency of our method, in particular in the presence of a skew.

1. Introduction

Monte Carlo simulation methods are used extensively by many financial institutions for the pricing of options. Therefore, there is an increasing need for numerical techniques which provide variance reduction. This paper focuses on the importance sampling method for variance reduction in the framework of stochastic volatility models.

A preliminary approximation for the expectation of interest is the main feature of the importance sampling technique. We provide two methods for obtaining this approximation. The first one, introduced in [3], is based on a small noise expansion in the volatility. It corresponds to a *regular perturbation* of the pricing Black–Scholes partial differential equation. The second one, introduced in this paper, is based on the fast mean-reverting stochastic volatility asymptotics described in [1]. It corresponds to a *singular perturbation* of the pricing partial differential equation. The leading-order term in this expansion is the Black–Scholes price with a constant *effective volatility*. We compare these two methods and show that the second one outperforms the first even when the volatility mean-reversion rate is of order one. In particular we show that, in the presence of a skew, the first correction greatly improves the simulation, unlike in the case of a regular perturbation as observed in [3].

The paper is organized as follows. In section 2 we present a class of stochastic volatility models which is often used in the pricing of options. For more details on option pricing under stochastic volatility we also refer to [7], the surveys [5], [4], or [6] for an example of a model with closed-form solution. In section 3 we recall the two classical approaches to pricing European options. In section 4 we give the general description of the importance sampling technique and its application to pricing an option. Small noise expansion is explained in section 5, while fast-mean reversion (FMR) asymptotics is described in section 6. Numerical results comparing the two methods of expansion are given in section 7. The appendix provides a brief review of the asymptotic expansion in fast-mean reverting stochastic volatility models.

2. A class of stochastic volatility models

Consider the price of a risky asset, X_t, which evolves according to the following stochastic differential equation:

$$dX_t = \mu X_t\, dt + \sigma(Y_t) X_t\, dW_t \qquad (2.1)$$

where μ represents a constant mean return rate, $\sigma(\cdot)$ represents the volatility which is driven by another stochastic process Y_t, and W_t represents a standard Brownian motion. We assume that the volatility is positive, bounded and bounded away from zero: $0 \leqslant \sigma_1 \leqslant \sigma(\cdot) \leqslant \sigma_2$ for two constants σ_1 and σ_2. The volatility is a function of an Ito process, Y_t, satisfying another stochastic differential equation driven by a second Brownian motion. In order to account for the leverage effect between stock price and volatility shocks we allow these two Brownian motions to be dependent.

The process Y_t which drives the volatility is commonly modelled as a mean-reverting process. The term 'mean reverting' refers to the fact that the process returns to the average value of its invariant distribution: the long-run distribution of the process. In terms of financial modelling, mean reverting often refers to a linear pull-back term in the drift of the volatility process. Usually, Y_t takes the following form:

$$dY_t = \alpha(m - Y_t)\,dt + \cdots + d\hat{Z}_t, \tag{2.2}$$

where $\hat{Z}_t$ is a Brownian motion correlated with W_t. The *rate of mean reversion* is represented by the parameter α and the mean level of the invariant distribution of Y_t is given by m. We consider here the simplest model which has this form: the Ornstein–Uhlenbeck process

$$dY_t = \alpha(m - Y_t)\,dt + \beta\,d\hat{Z}_t, \tag{2.3}$$

where $\beta > 0$ is a constant and $\hat{Z}_t$ is a Brownian motion expressed as:

$$\hat{Z}_t = \rho W_t + \sqrt{1-\rho^2}\,Z_t,$$

where Z_t is a standard Brownian motion independent of W_t. The parameter $\rho \in (-1, 1)$ is the constant instantaneous correlation coefficient between $\hat{Z}_t$ and W_t.

The invariant distribution of Y_t is the Gaussian distribution $\mathcal{N}(m, \beta^2/2\alpha)$. Denoting its variance by $\nu^2 = \beta^2/2\alpha$ and substituting for β in (2.3) we get

$$dY_t = \alpha(m - Y_t)\,dt + \nu\sqrt{2\alpha}\,d\hat{Z}_t. \tag{2.4}$$

In the following we assume that m and ν are fixed quantities and we will be interested in the two regimes $\alpha \to 0$ (small noise) and $\alpha \to +\infty$ (fast mean reversion). A volatility function $\sigma(\cdot)$ will be chosen later on in order to perform numerical simulations. As we shall see the results obtained with fast mean-reversion expansion are robust with respect to that choice.

3. Pricing European options

For simplicity we deal with a European call option which is a contract that gives its holder the right, but not the obligation, to buy at *maturity* T one unit of the underlying asset for a predetermined *strike price* K. The value of this call option at maturity, its *payoff*, is given by

$$\phi(X_T) = (X_T - K)^+ = \begin{cases} X_T - K & \text{if} \quad X_T > K \\ 0 & \text{if} \quad X_T \leqslant K. \end{cases}$$

We now summarize two approaches to the problem of pricing such an option.

3.1. Equivalent martingale measure approach

In our class of models, the price X_t of the underlying evolves under the 'physical' measure according to the following system of equations:

$$\begin{aligned} dX_t &= \mu X_t\,dt + \sigma(Y_t)X_t\,dW_t \\ dY_t &= \alpha(m - Y_t)\,dt + \nu\sqrt{2\alpha}\left(\rho\,dW_t + \sqrt{1-\rho^2}\,dZ_t\right). \end{aligned} \tag{3.1}$$

Let $\mathbb{P}$ denote the probability measure related to the Brownian vector (W_t, Z_t). It would seem reasonable that the price of an option is the expected discounted payoff under the probability measure $\mathbb{P}$. However, the discounted price $\tilde{X}_t = e^{-rt}X_t$, where r represents a constant instantaneous interest rate for borrowing or lending money, is not a martingale under this measure if $\mu \neq r$. A *no-arbitrage* argument shows that this expected discounted payoff should be computed under an equivalent probability $\mathbb{P}^* \sim \mathbb{P}$ under which the discounted price is a martingale. Due to the presence of the second source of randomness Z_t, this equivalent martingale measure is not unique. As detailed in [1], we 'parametrize' the problem by a *market price of volatility risk* $\gamma(y)$ which we assume to be bounded and dependent only on y. The evolution of the price X_t under the risk-neutral measure $\mathbb{P}^{*(\gamma)}$, chosen by the market, is given by the following system of equations:

$$\begin{aligned} dX_t &= rX_t\,dt + \sigma(Y_t)X_t\,dW_t^* \\ dY_t &= \left[\alpha(m - Y_t) - \nu\sqrt{2\alpha}\,\Lambda(y)\right]dt \\ &\quad + \nu\sqrt{2\alpha}\left(\rho\,dW_t^* + \sqrt{1-\rho^2}\,dZ_t^*\right), \end{aligned} \tag{3.2}$$

where

$$\Lambda(y) = \frac{\rho(\mu - r)}{\sigma(y)} + \gamma(y)\sqrt{1-\rho^2} \tag{3.3}$$

accounts for the market prices of risk and W_t^* and Z_t^* are independent standard Brownian motions. The price of an option depends upon the volatility risk premium factor γ and its value, $P(t, X_t, Y_t)$, is computed under the risk-neutral measure, $\mathbb{P}^{*(\gamma)}$, as

$$P(t, x, y) = \mathbb{E}^{*(\gamma)}\{e^{-r(T-t)}\phi(X_T) | X_t = x, Y_t = y\}. \tag{3.4}$$

Notice that this price depends also on the current volatility level y which is not directly observable.

3.2. Partial differential equation approach

By the Feynman–Kac formula, the pricing function given by (3.4) satisfies the following partial differential equation with two space dimensions:

$$\begin{aligned} &\frac{\partial P}{\partial t} + \frac{1}{2}\sigma^2(y)x^2\frac{\partial^2 P}{\partial x^2} + \rho\nu\sqrt{2\alpha}\,x\sigma(y)\frac{\partial^2 P}{\partial x\partial y} + \nu^2\alpha\,\frac{\partial^2 P}{\partial y^2} \\ &\quad + r\left(x\frac{\partial P}{\partial x} - P\right) + \left[(\alpha(m - y)) - \nu\sqrt{2\alpha}\,\Lambda(y)\right]\frac{\partial P}{\partial y} \\ &\quad = 0, \end{aligned} \tag{3.5}$$

where $\Lambda(y)$ is given by (3.3). In order to find $P(t, x, y)$, this partial differential equation (PDE) is solved backward in time with the *terminal condition* $P(T, x, y) = \phi(x)$ which is $(x-K)^+$ in the case of a call option. We introduce the following convenient operator notation:

$$\mathcal{L}_0 = \nu^2 \frac{\partial^2}{\partial y^2} + (m-y)\frac{\partial}{\partial y} \tag{3.6}$$

$$\mathcal{L}_1 = \rho\nu\sqrt{2}\,x\sigma(y)\frac{\partial^2}{\partial x \partial y} - \nu\sqrt{2}\,\Lambda(y)\frac{\partial}{\partial y} \tag{3.7}$$

$$\mathcal{L}_2 = \frac{\partial}{\partial t} + \frac{1}{2}\sigma^2(y)x^2\frac{\partial^2}{\partial x^2} + r\left(x\frac{\partial}{\partial x} - \cdot\right), \tag{3.8}$$

where

- $\alpha\mathcal{L}_0$ is the infinitesimal generator of the OU process Y_t.
- $\mathcal{L}_1$ contains the mixed partial derivative due to the correlation ρ between W^* and Z^*. It also contains the first-order derivative with respect to y due to the market prices of risk.
- $\mathcal{L}_2$ is the Black–Scholes operator with volatility $\sigma(y)$, also denoted by $\mathcal{L}_{\mathrm{BS}(\sigma(y))}$.

Equation (3.5) may be written in the compact form

$$\left(\alpha\mathcal{L}_0 + \sqrt{\alpha}\mathcal{L}_1 + \mathcal{L}_2\right) P = 0, \tag{3.9}$$

to be solved with the payoff terminal condition at maturity T.

4. Importance sampling for diffusions

In this section a description of the importance sampling variance reduction technique for diffusions is given. The reader is referred to [8] for more details.

4.1. General description for diffusion models

Let $(V_t)_{0\leqslant t\leqslant T}$ be an n-dimensional stochastic process which evolves as follows:

$$\mathrm{d}V_t = b(t, V_t)\,\mathrm{d}t + a(t, V_t)\,\mathrm{d}\eta_t, \tag{4.1}$$

where η_t is a standard n-dimensional $\mathbb{P}$-Brownian motion and $b(\cdot,\cdot) \in \mathbb{R}^n$, $a(\cdot,\cdot) \in \mathbb{R}^{n\times n}$ which satisfy the usual regularity and boundedness assumptions to ensure existence and uniqueness of the solution. Given a real function $\phi(v)$ with polynomial growth we define the following function $u(t, v)$:

$$u(t, v) = \mathbb{E}\{\phi(V_T)|V_t = v\}.$$

A Monte Carlo simulation consists of approximating $u(t, v)$ in the following manner:

$$u(t, v) \approx \frac{1}{N}\sum_{k=1}^{N}\phi(V_T^{(k)}), \tag{4.2}$$

where $(V_\cdot^{(k)}, k = 1, \ldots, N)$ are independent realizations of the process V for $t \leqslant \cdot \leqslant T$ and $V_t^{(k)} = v$.

There is an alternative way to construct a Monte Carlo approximation of $u(t, v)$. Given a square integrable $\mathbb{R}^n$-valued, η-adapted process of the form $h(t, V_t)$, we consider the following process Q_t:

$$Q_t = \exp\left\{\int_0^t h(s, V_s)\cdot \mathrm{d}\eta_s + \tfrac{1}{2}\int_0^t \|h(s, V_s)\|^2\,\mathrm{d}s\right\}.$$

If $\mathbb{E}(Q_t^{-1}) = 1$, then $(Q_t)_{0\leqslant t\leqslant T}$ is a positive martingale and a new probability measure, $\tilde{\mathbb{P}}$, may be defined by the density

$$\frac{\mathrm{d}\tilde{\mathbb{P}}}{\mathrm{d}\mathbb{P}} = (Q_T)^{-1}.$$

With respect to this new measure, $u(t, v)$ can be written

$$u(t, v) = \tilde{\mathbb{E}}\{\phi(V_T)Q_T|V_t = v\}. \tag{4.3}$$

By Girsanov's theorem, the process $(\tilde{\eta}_t)_{0\leqslant t\leqslant T}$ defined by $\tilde{\eta}_t = \eta_t + \int_0^t h(s, V_s)\,\mathrm{d}s$ is a standard Brownian motion under the new measure $\tilde{\mathbb{P}}$. In terms of the Brownian motion $\tilde{\eta}$, the processes V_t and Q_t can be rewritten as

$$\mathrm{d}V_t = (b(t, V_t) - a(t, V_t)h(t, V_t))\,\mathrm{d}t + a(t, V_t)\,\mathrm{d}\tilde{\eta}_t \tag{4.4}$$

$$Q_t = \exp\left\{\int_0^t h(s, V_s)\cdot \mathrm{d}\tilde{\eta}_s - \tfrac{1}{2}\int_0^t \|h(s, V_s)\|^2\,\mathrm{d}s\right\} \tag{4.5}$$

which will be used in the simulations for the approximation of (4.3) by

$$u(t, v) \approx \frac{1}{N}\sum_{k=1}^{N}\phi(V_T^{(k)})Q_T^{(k)}. \tag{4.6}$$

The *variance reduction method* consists of determining functions $h(t, v)$ which lead to a smaller variance for the Monte Carlo approximation given in (4.6) than the variance for (4.2).

Applying Ito's formula to $u(t, V_t)Q_t$ and using the Kolmogorov backward equation for $u(t, v)$ one gets

$$\begin{aligned}\mathrm{d}(u(t, V_t)Q_t) &= u(t, V_t)Q_t h(t, V_t)\cdot \mathrm{d}\tilde{\eta}_t \\ &\quad + Q_t a^T(t, V_t)\nabla u(t, V_t)\cdot \mathrm{d}\tilde{\eta}_t \\ &= Q_t(a^T\nabla u + uh)(t, V_t)\cdot \mathrm{d}\tilde{\eta}_t\end{aligned}$$

where a^T denotes the transpose of a, and ∇u the gradient of u with respect to the state variable v.

In order to obtain $u(0, v)$, for instance, one can integrate between 0 and T and deduce

$$u(T, V_T)Q_T = u(0, V_0)Q_0 + \int_0^T Q_t(a^T\nabla u + uh)(t, V_t)\cdot \mathrm{d}\tilde{\eta}_t,$$

which reduces to

$$\phi(V_T)Q_T = u(0, v) + \int_0^T Q_t(a^T\nabla u + uh)(t, V_t)\cdot \mathrm{d}\tilde{\eta}_t.$$

Therefore the variances in the two Monte Carlo simulations (4.2) and (4.6), are given by

$$\mathrm{Var}_{\tilde{\mathbb{P}}}(\phi(V_T)Q_T) = \tilde{\mathbb{E}}\left\{\int_0^T Q_t^2\|a^T\nabla u + uh\|^2\,\mathrm{d}t\right\}$$

$$\mathrm{Var}_{\mathbb{P}}(\phi(V_T)) = \mathbb{E}\left\{\int_0^T \|a^T\nabla u\|^2\,\mathrm{d}t\right\}.$$

If $u(t, v)$ were known, then the problem would be solved and the optimal choice for h, which gives a zero variance, would be

$$h = -\frac{1}{u}a^T\nabla u. \tag{4.7}$$

In other words the ith component of h is given by

$$h_i(t,v) = -\frac{1}{u(t,v)}\sum_{j=1}^{n} a_{j,i}(t,v)\frac{\partial u}{\partial v_j}(t,v).$$

The main idea is to use an approximation for the unknown u in the previous formula which gives a function h such that Girsanov's theorem applies and the variance of Q_t can be controlled. Before doing so, we first rewrite (4.7) in the case of stochastic volatility models described in section 2.

4.2. Application to stochastic volatility models

We apply the change of measure technique to the class of models described by (3.2) and used for computing European call options. In matrix form the evolution under the risk-neutral measure $\mathbb{P}^*$ is given by

$$\mathrm{d}V_t = b(V_t)\,\mathrm{d}t + a(V_t)\,\mathrm{d}\eta_t, \tag{4.8}$$

where we have set

$$\eta_t = \begin{pmatrix} W_t^* \\ Z_t^* \end{pmatrix}, \qquad V_t = \begin{pmatrix} X_t \\ Y_t \end{pmatrix},$$

and

$$a(v) = \begin{pmatrix} x\sigma(y) & 0 \\ \nu\rho\sqrt{2\alpha} & \nu\sqrt{2\alpha(1-\rho^2)} \end{pmatrix},$$

$$b(v) = \begin{pmatrix} rx \\ \alpha(m-y) - \nu\sqrt{2\alpha}\,\Lambda(y) \end{pmatrix}.$$

The price of a call option at time 0 is computed by

$$P(0,v) = \mathbb{E}^*\{\mathrm{e}^{-rT}\phi(v)|V_0 = v\},$$

where $v=(x,y)$ and $\phi(v) = (x-K)^+$.

We now apply the importance sampling technique described in section 4.1.

Define $\tilde{\eta}_t = \eta_t + \int_0^t h(s,V_s)\,\mathrm{d}s$ which is a Brownian motion under the probability $\tilde{\mathbb{P}}^*$ which admits the density Q_T^{-1} as described in section 4.1.

Under the new measure, the price of the call option at time 0 is then computed by

$$P(0,v) = \tilde{\mathbb{E}}^*\{\mathrm{e}^{-rT}\phi(v)Q_T|V_0 = v\}. \tag{4.9}$$

By (4.7), if $P(t,x,y)$ were known, the optimal choice for h would be

$$h(t,x,y) = -\frac{1}{P(t,x,y)}\begin{pmatrix} x\sigma(y) & \nu\rho\sqrt{2\alpha} \\ 0 & \nu\sqrt{1-\rho^2}\sqrt{2\alpha} \end{pmatrix} \times \begin{pmatrix} \frac{\partial P}{\partial x}(t,x,y) \\ \frac{\partial P}{\partial y}(t,x,y) \end{pmatrix}. \tag{4.10}$$

Once we have found an approximation of P using small noise expansion or fast mean-reversion expansion, then we may determine h in order to approximate (4.9) via Monte Carlo simulations (4.6) under the evolution (4.4), (4.5).

5. Small noise expansion

In the implementation of the importance sampling variance reduction technique, the approximation of $P(t,x,y)$ proposed in [3] is obtained by performing a regular perturbation of the pricing PDE (3.9) given by

$$\left(\alpha\mathcal{L}_0 + \sqrt{\alpha}\mathcal{L}_1 + \mathcal{L}_2\right)P = 0,$$

with the terminal condition $P(T,x,y) = (x-K)^+$. If $\alpha = 0$, then this PDE becomes

$$\mathcal{L}_2 P = 0.$$

Since $\mathcal{L}_2$ is simply the Black–Scholes operator with constant volatility $\sigma(y)$, then an approximation $P_{\mathrm{BS}(\sigma(y))}$ of P is given by the Black–Scholes formula

$$P_{\mathrm{BS}(\sigma(y))}(t,x) = xN(d_1) - K\mathrm{e}^{-r(T-t)}N(d_2),$$

where

$$N(d) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{d}\mathrm{e}^{-z^2/2}\,\mathrm{d}z,$$

$$d_1 = \frac{\ln(x/K) + (r+\sigma^2(y)/2)(T-t)}{\sigma(y)\sqrt{T-t}},$$

$$d_2 = d_1 - \sigma(y)\sqrt{T-t}.$$

The function $P_{\mathrm{BS}(\sigma(y))}(t,x)$ is the first term in the small noise expansion of $P(t,x,y)$ as $\alpha \to 0$ or, in other words, when volatility is slowly varying and, in the limit, Y_t being 'frozen' at its initial point y. A complete proof of the expansion result with higher-order terms is given in [3] as well as numerical results showing that the important gain in variance reduction is obtained by using the leading-order term $P_{\mathrm{BS}(\sigma(y))}$ alone as an approximation of P. In that case, from (4.10) with $\alpha = 0$, h takes the following form:

$$h(t,x,y) = -\frac{1}{P_{\mathrm{BS}(\sigma(y))}(t,x)}\begin{pmatrix} x\sigma(y)\frac{\partial P_{\mathrm{BS}(\sigma(y))}}{\partial x}(t,x) \\ 0 \end{pmatrix}. \tag{5.1}$$

Recall that the delta $\partial P_{\mathrm{BS}(\sigma(y))}/\partial x$ is given by $N(d_1)$ computed with $\sigma(y)$. In particular it is bounded and h is such that Girsanov's theorem applies and Q_t has a finite variance.

6. Fast mean-reversion expansion

Since α represents the rate of mean reversion, then *fast mean reversion* refers to α being large. This can also be interpreted as $1/\alpha$, the intrinsic decorrelation time in volatility, being small. We refer to [1] for more details. In order to implement the importance sampling variance reduction technique for fast mean reversion, an approximation of $P(t,x,y)$ is obtained by performing a *singular perturbation* of the pricing PDE (3.9). We briefly recall in appendix A.2 that, as α becomes large, $P(t,x,y)$ has a limit $P_{\mathrm{BS}(\bar{\sigma})}(t,x)$ which is the Black–Scholes price of the call option with a *constant effective volatility* $\bar{\sigma}$. In particular $P_{\mathrm{BS}(\bar{\sigma})}$ does not depend on y and is given by

$$P_{\mathrm{BS}(\bar{\sigma})}(t,x) = xN(d_1) - K\mathrm{e}^{-r(T-t)}N(d_2), \tag{6.1}$$

where

$$N(d) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{d} e^{-z^2/2}\, dz,$$

$$d_1 = \frac{\ln(x/K) + (r + \bar{\sigma}^2/2)(T-t)}{\bar{\sigma}\sqrt{T-t}}$$

and

$$d_2 = d_1 - \bar{\sigma}\sqrt{T-t},$$

and $\bar{\sigma}$ is the effective volatility presented in appendix A.1. At this level of approximation the choice of h is given by (4.10) with P being replaced by $P_{BS(\bar{\sigma})}$

$$h(t,x,y) = -\frac{1}{P_{BS(\bar{\sigma})}(t,x)} \begin{pmatrix} \sigma(y)x \frac{\partial P_{BS(\bar{\sigma})}}{\partial x}(t,x) \\ 0 \end{pmatrix}. \quad (6.2)$$

Observe that it is extremely important that $\partial P_{BS(\bar{\sigma})}/\partial y = 0$ in order to cancel the diverging terms in $\sqrt{\alpha}$ appearing in the second column of a^T in (4.10). With this choice of h, as in the previous section, Girsanov's theorem applies and the variance of Q_t is finite. The numerical results presented in the next section will show that Monte Carlo simulations using this approximation already outperform the ones using the small noise approximation. It is even possible to improve the method greatly by including the first correction in the approximation. As recalled in appendix A.2, the first correction of order $1/\sqrt{\alpha}$ in the fast mean-reversion expansion is also independent of y and is given by

$$-(T-t)\left(V_2 x^2 \frac{\partial^2 P_{BS(\bar{\sigma})}}{\partial x^2} + V_3 x^3 \frac{\partial^3 P_{BS(\bar{\sigma})}}{\partial x^3}\right),$$

where V_2 and V_3 are constants of order $1/\sqrt{\alpha}$ depending on the model parameters as described in appendix A.2. One of the great advantages of this approach is that these two constants V_2 and V_3 can be calibrated from the observed skew as shown in [1] and discussed in appendix A.3.
Hence, we approximate $P(t,x,y)$ by the corrected Black–Scholes price $P_{FMR}(t,x)$ given by

$$P_{FMR} = P_{BS(\bar{\sigma})} - (T-t)\left(V_2 x^2 \frac{\partial^2 P_{BS(\bar{\sigma})}}{\partial x^2} + V_3 x^3 \frac{\partial^3 P_{BS(\bar{\sigma})}}{\partial x^3}\right). \quad (6.3)$$

From (4.10), and again using $\partial P_{FMR}/\partial y = 0$, we deduce that h takes the form:

$$h_{FMR}(t,x,y) = -\frac{1}{P_{FMR}(t,x)} \begin{pmatrix} \sigma(y)x \frac{\partial P_{FMR}}{\partial x}(t,x) \\ 0 \end{pmatrix}. \quad (6.4)$$

The presence of higher-order derivatives of the Black–Scholes price of call option introduces a problem when approaching maturity close to the strike price. For a detailed analysis of this issue we refer to [2]. In the following numerical experiments we have introduced a cutoff which consists of choosing $h = 0$ near maturity so that Girsanov's theorem applies again and the variance of Q_t is controlled by this cutoff parameter. The important part of the correction comes from the third derivative term and since, as shown in appendix A.3, V_3 is proportional to ρ, it will improve the Monte Carlo simulation only if there is a leverage effect $\rho \neq 0$.

Table 1. The variance for each method of approximation.

α	Basic Monte Carlo	$P_{BS(\sigma(y))}$	$P_{BS(\bar{\sigma})}$	P_{FMR}
0.5	0.0164	0.0026	0.0028	0.0021
1	0.0205	0.0046	0.0044	0.0013
5	0.0232	0.0081	0.0036	0.0012
10	0.0237	0.0083	0.0028	0.0008
25	0.0257	0.0115	0.0010	0.0007
50	0.0288	0.0150	0.0007	0.0006
100	0.0319	0.0184	0.0004	0.0003

7. Numerical results

In this section we present numerical results comparing both methods of expansion. The experiments are based on the following parameters of the model given in (3.2)

$$r = 0.1, \qquad \sigma(y) = \exp(y),$$

and

$$m = -2.6, \qquad \nu = 1, \qquad \Lambda(y) = 0, \qquad \rho = -0.3.$$

Large values of $|Y_t|$ have been cut off so that σ remains bounded. This does not affect the model significantly. With this choice of $\sigma(y)$, m and ν we get the value $\bar{\sigma} = 0.2$ by (A.5).

We have used the following initial values:

$$X_0 = 110, \qquad Y_0 = -2.32,$$

for the call option at $K = 100$ and $T = 1$. Since both methods are characterized by the rate of mean reversion, we present results for various values of α ranging from slow mean reversion $\alpha = 0.5$ to fast mean reversion $\alpha = 100$.

We use a Euler scheme to approximate the diffusion process V_t used in the Monte Carlo simulations (4.2) or (4.6). The choice of h varies with the method described in sections 5 or 6. In addition, the time step is 10^{-3} and the number of realizations used is 10 000. Results are presented in table 1.

Basic Monte Carlo refers to computing the price under the measure $\mathbb{P}^*$, while $P_{BS(\sigma(y))}$ refers to using a small noise expansion of $P(t,x,y)$ when determining the process h in (5.1). The quantities $P_{BS(\bar{\sigma})}$ and P_{FMR} correspond to using a FMR expansion of $P(t,x,y)$ when computing the process h in (6.2) or (6.4).

Figures 1 and 2 present the results of our Monte Carlo simulations as a function of the number of realizations. The two illustrations given are for $\alpha = 1$ and 10. It is clear from the table and figures that the basic Monte Carlo estimator performs extremely poorly when compared to the other three estimators. Also, notice that when $\alpha = 1$ the variance for small noise expansion and zero-order fast mean-reversion expansion are approximately the same; however, when the first correction is added to the approximation we obtain a greater reduction in the variance. Additionally, when the rate of mean reversion is extremely large, the order zero and order one approximations are about the same.

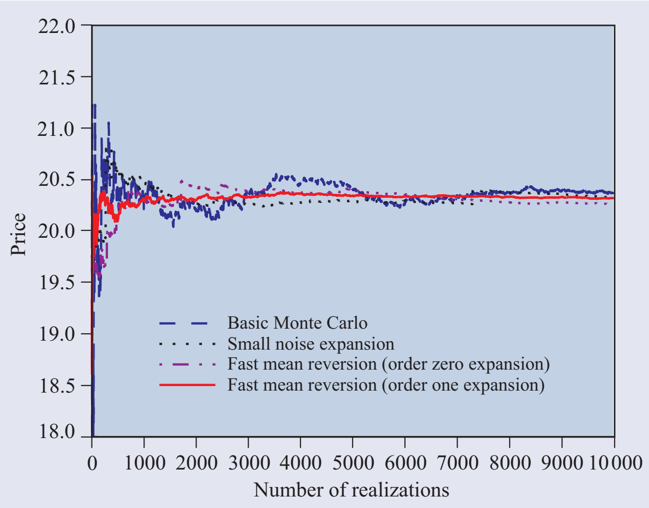

Figure 1. Monte Carlo simulations with a rate of mean reversion of $\alpha = 1$.

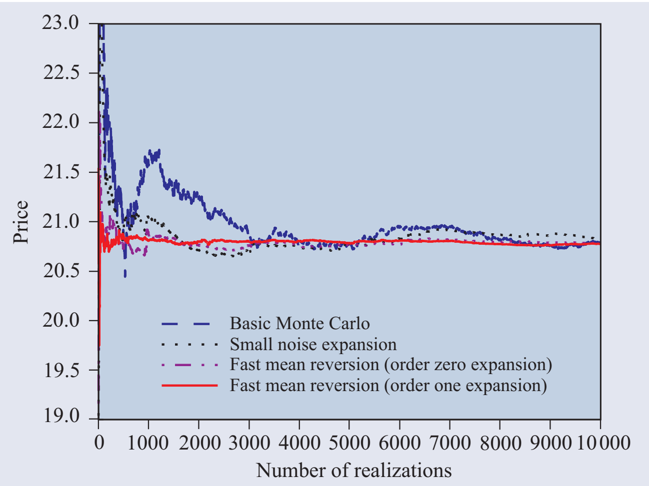

Figure 2. Monte Carlo simulations with a rate of mean reversion of $\alpha = 10$.

8. Generalizations

8.1. Multidimensional case

Monte Carlo methods become competitive against PDE methods in particular when the number of underlyings on which the option is written is not small. Furthermore the number of state variables is basically doubled in the context of a stochastic volatility matrix. Therefore variance reduction techniques become extremely important. Fast mean-reversion asymptotics work as well in this situation as shown in [1] (ch 10, section 6) where an effective volatility matrix is introduced in order to compute the approximated price used in changing the measure. We recommend such a method when the constant volatility problem can be solved by PDE methods and the corresponding stochastic volatility problem requires Monte Carlo simulations due to the large number of state variables.

8.2. Jumps

Jumps can be introduced in the model in different ways. For instance one can consider jumps in volatility. In that case the fast mean reversion can be performed as well as shown in [1] (ch 10, section 3). This leads again to an effective volatility used to compute the approximated price. Another way to introduce jumps is to consider possible jumps in the underlying itself, combined with stochastic volatility. Monte Carlo methods are well adapted to this situation. Fast mean-reversion asymptotics can be performed, leading to a model with jumps and constant effective volatility. If prices can be computed efficiently within this simplified model then our variance reduction technique can be applied. This will be the topic of a future investigation.

8.3. Barrier and other options

We are presently working on the implementation of our method for pricing barrier options in the context of stochastic volatility. Fast mean-reversion asymptotics have been developed in [1] (ch 8, section 2). The first term in the approximation is the usual constant volatility barrier price given explicitly by the method of images. The correction is not given explicitly but as an integral involving the density of hitting times. This is a promising work in progress. The case of American options will also be investigated.

9. Conclusion

We have shown that fast mean-reversion asymptotics can be used in importance sampling variance reduction techniques used in Monte Carlo computations of options prices in the context of stochastic volatility. Extensive numerical experiments for European calls show that these asymptotics are very efficient even when volatility is not fast mean reverting. These results are summarized in the table presented in section 7. In particular, in the presence of a skew, the first correction is very efficient in reducing the variance. This is in contrast with another approach based on small noise expansion. Our work in progress indicates that the method is also very efficient for other types of options.

Appendix

A.1. Effective volatility

The process Y_t has an invariant distribution which admits the density $\Phi(y)$ obtained by solving the adjoint equation

$$\mathcal{L}_0^*\Phi = 0,$$

where $\mathcal{L}_0^*$ denotes the adjoint of the infinitesimal generator $\mathcal{L}_0$ given by (3.6). In the case of the Ornstein–Uhlenbeck process, which we consider in this paper, the invariant distribution is $\mathcal{N}(m, \nu^2)$ and the density is explicitly given by

$$\Phi(y) = \frac{1}{\sqrt{2\pi\nu^2}} \exp\left(-\frac{(y-m)^2}{2\nu^2}\right).$$

Let $\langle\cdot\rangle$ denote the average with respect to this invariant distribution

$$\langle g\rangle = \int_{-\infty}^{\infty} g(y)\Phi(y)\,\mathrm{d}y.$$

Given a bounded function g, by the ergodic theorem, the long-time average of $g(Y_t)$ is close to the average with respect to the invariant distribution

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t g(Y_s)\,\mathrm{d}s = \langle g\rangle.$$

In our case the 'real time' for the process Y_t is the product αt and long-time behaviour is the same in distribution as the large mean-reversion rate, and therefore

$$\frac{1}{t}\int_0^t g(Y_s)\,\mathrm{d}s \approx \langle g\rangle,$$

for α large and any fixed $t > 0$. In particular, in the context of stochastic volatility models, we consider the *mean-square-time-averaged volatility* $\overline{\sigma^2}$ defined by

$$\overline{\sigma^2} = \frac{1}{T-t}\int_t^T \sigma^2(Y_s)\,\mathrm{d}s.$$

The above result shows that, for large α

$$\overline{\sigma^2} \approx \langle\sigma^2\rangle \equiv \bar{\sigma}^2, \tag{A.1}$$

which defines the constant *effective volatility* $\bar{\sigma}$. This quantity is easily estimated from the observed fluctuations in returns. We refer to [1] for more details.

A.2. Fast mean-reverting stochastic volatility asymptotics

Using the notation of section 3.2, the price $P(t, X_t, Y_t)$ of a European call option depends on the current values of the underlying and volatility level. The function $P(t, x, y)$ is obtained as the solution of the pricing PDE (3.5) with the appropriate terminal condition $P(T, x, y) = (x - K)^+$. This equation takes the form (3.9)

$$\left(\alpha\mathcal{L}_0 + \sqrt{\alpha}\mathcal{L}_1 + \mathcal{L}_2\right) P = 0,$$

with the operator notation (3.6)–(3.8). Fast mean reversion corresponds to α large and therefore to a singular perturbation of this equation due to the diverging terms, keeping the time derivative in $\mathcal{L}_2$ of order one.

Expanding P in powers of $1/\sqrt{\alpha}$

$$P = P_0 + \frac{1}{\sqrt{\alpha}}P_1 + \frac{1}{\alpha}P_2 + \frac{1}{\alpha\sqrt{\alpha}}P_3 + \cdots,$$

it is shown in [1] that $P_0(t, x) = P_{\mathrm{BS}(\bar{\sigma})}(t, x)$ is the solution of the Black–Scholes equation

$$\frac{\partial P_0}{\partial t} + \frac{1}{2}\bar{\sigma}^2 x^2\frac{\partial^2 P_0}{\partial x^2} + r\left(x\frac{\partial P_0}{\partial x} - P_0\right) = 0, \tag{A.2}$$

with constant effective volatility $\bar{\sigma}$ given by (A.1) and terminal condition $P_0(T, x) = (x - K)^+$.

In addition, the first correction $\tilde{P}_1(t, x) \equiv \frac{1}{\sqrt{\alpha}}P_1(t, x)$ is also independent of y. It is the solution of the same Black–Scholes equation but with a zero terminal condition and a source. It is given explicitly by

$$\tilde{P}_1 = -(T-t)\left(V_2 x^2\frac{\partial^2 P_0}{\partial x^2} + V_3 x^3\frac{\partial^3 P_0}{\partial x^3}\right),$$

where V_2 and V_3 are two constants of order $1/\sqrt{\alpha}$ related to the model parameters by

$$V_2 = \frac{1}{\nu\sqrt{2\alpha}}\langle(-2\rho R + S)(\sigma^2 - \langle\sigma^2\rangle)\rangle \tag{A.3}$$

$$V_3 = \frac{-\rho}{\nu\sqrt{2\alpha}}\langle R(\sigma^2 - \langle\sigma^2\rangle)\rangle, \tag{A.4}$$

where R and S denote antiderivatives of σ and Λ, respectively (see [1]).

A.3. Calibration of parameters

In [1] it is shown how to calibrate V_2 and V_3 from the implied volatility skew. This is the natural way to use the fast mean-reversion expansion when the rate of mean reversion is in fact large, as demonstrated in the case of the S&P 500 index for instance. Here we are in a different situation where a model of stochastic volatility has been chosen with a rate of mean reversion which may not be large but rather of order one. In particular a function $\sigma(y)$ has been prescribed and the other model parameters have been estimated one way or another. The goal is to compute derivative prices by Monte Carlo simulations under this model. In order to use the FMR variance reduction technique one has to compute from the model parameters $(r, m, \nu, \alpha, \rho, \Lambda)$ the three quantities $\bar{\sigma}$, V_2 and V_3 using

$$\bar{\sigma}^2 = \frac{1}{\sqrt{2\pi\nu^2}}\int_{-\infty}^{\infty}\sigma^2(y)\exp\left(-\frac{(y-m)^2}{2\nu^2}\right)\mathrm{d}y, \tag{A.5}$$

for $\bar{\sigma}$ and (A.3), (A.4) for V_2, V_3. We then compute successively $P_{\mathrm{BS}(\bar{\sigma})}$ and P_{FMR} using (6.1) and (6.3).

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References

[1] Fouque J P, Papanicolaou G and Sircar K R 2000 *Derivatives in Financial Markets with Stochastic Volatility* (Cambridge: Cambridge University Press)

[2] Fouque J P, Papanicolaou G, Sircar K R and Solna K 2001 Singular perturbations in option pricing *Preprint*

[3] Fournie E, Lebuchoux J and Touzi N 1997 Small noise expansion and importance sampling *Asymptotic Anal.* **14** 361–76

[4] Frey R 1996 Derivative asset analysis in models with level-dependent and stochastic volatility *CWI Quarterly* **10** 1–34

[5] Ghysels E, Harvey A and Renault E 1996 Stochastic volatility *Statistical Methods in Finance (Handbook of Statistics vol 14)* ed G Maddala and C Rao (Amsterdam: North-Holland) ch 5 pp 119–91

[6] Heston S 1993 A closed-form solution for options with stochastic volatility with applications to bond and currency options *Rev. Financial Studies* **6** 327–43

[7] Hull J and White A 1987 The pricing of options on assets with stochastic volatilities *J. Finance* **XLII** 281–300

[8] Newton N 1994 Variance reduction for simulated diffusions *SIAM J. Appl. Math.* **54** 1780–805